

ASSIGNMENT 1

September 3, 2008

- Using the Trichotomy Law prove that if a and b are real numbers then one and only one of the following is possible: $a < b$, $a = b$, or $a > b$.
- We define the *absolute value* of a real number a by

$$|a| = \begin{cases} a, & a \geq 0 \\ -a, & a \leq 0 \end{cases}$$

Prove the following:

- $|a + b| \leq |a| + |b|$.
 - $|xy| = |x| \cdot |y|$.
 - $\left| \frac{1}{x} \right| = \frac{1}{|x|}$, if $x \neq 0$.
 - $\left| \frac{x}{y} \right| = \frac{|x|}{|y|}$, if $y \neq 0$.
 - $|x - y| \leq |x| + |y|$.
 - $|x| - |y| \leq |x - y|$.
- The fact that $a^2 \geq 0$ for all real numbers a has tremendous implications. The most widely used of all inequalities is the *Schwarz inequality*:

$$x_1y_1 + x_2y_2 \leq \sqrt{x_1^2 + x_2^2} \sqrt{y_1^2 + y_2^2}$$

Do ONE of the following:

- Prove the Schwarz inequality by using $2xy \leq x^2 + y^2$ (how is this derived?) with

$$x = \frac{x_i}{\sqrt{x_1^2 + x_2^2}}, \quad y = \frac{y_i}{\sqrt{y_1^2 + y_2^2}}$$

first for $i = 1$ and then for $i = 2$.

- Prove the Schwarz inequality by first proving that

$$(x_1^2 + x_2^2)(y_1^2 + y_2^2) = (x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2.$$

- Prove the following formulæ by induction

- $1^2 + 2^2 + \cdots + n^2 = \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$

- $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$

- Find a formula for

$$(a) \sum_{i=1}^n (2i-1) = 1 + 3 + 5 + 7 + \cdots + (2n-1)$$

$$(b) \sum_{i=1}^n (2i-1)^2 = 1^2 + 3^2 + 5^2 + 7^2 + \cdots + (2n-1)^2$$

Hint: What do these expressions have to do with $1+2+3+\dots+2n$ and $1^2+2^2+3^2+\dots+(2n)^2$?

6. The formula for $1^2 + \cdots + n^2$ may be derived as follows. Start with the formula:

$$(k+1)^3 - k^3 = 3k^2 + 3k + 1.$$

Writing this formula for $k = 1, \dots, n$ and adding, we get

$$\begin{aligned} 2^3 - 1^3 &= 3 \cdot 1^2 + 3 \cdot 1 + 1 \\ 3^3 - 2^3 &= 3 \cdot 2^2 + 3 \cdot 2 + 1 \\ 4^3 - 3^3 &= 3 \cdot 3^2 + 3 \cdot 3 + 1 \\ &\vdots \\ (n+1)^3 - n^3 &= 3 \cdot n^2 + 3 \cdot n + 1 \\ \hline (n+1)^3 - 1 &= 3[1^2 + \cdots + n^2] + 3[1 + \cdots + n] + n \end{aligned}$$

Solving for the first term on the right we have:

$$\begin{aligned} 3[1^2 + \cdots + n^2] &= (n+1)^3 - 1 - 3[1 + \cdots + n] - n \\ 1^2 + \cdots + n^2 &= \frac{n(n+1)(2n+1)}{6} \end{aligned}$$

Use this method to find:

- (a) $1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3$
 (b) $1^4 + 2^4 + 3^4 + 4^4 + \cdots + n^4$
 (c) $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)}$