MATH 6101 090 Solutions Assignment 2

1. We will use the following two facts in this problem:

$$1+a+a^2+a^3+\cdots=\sum_{n=0}^{\infty}a^n=\frac{1}{1-a}\text{ if }-1<a<1$$

$$1+a+a^2+a^3+\cdots a^n=\frac{1-a^{n+1}}{1-a}\text{ if }a\neq 1$$

a) $1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\cdots=\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n=\frac{1}{1-\frac{1}{4}}=\frac{4}{3}$

b) $1-\frac{1}{4}+\frac{1}{16}-\frac{1}{64}+\cdots=\sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^n=\frac{1}{1-(-\frac{1}{4})}=\frac{4}{5}$

c) $1+\frac{3}{5}+\frac{9}{25}+\frac{27}{125}+\cdots=\sum_{n=0}^{\infty}\left(\frac{3}{5}\right)^n=\frac{1}{1-\frac{3}{5}}=\frac{5}{2}$

d) $3-\frac{9}{4}+\frac{27}{16}-\frac{81}{64}+\cdots=\sum_{n=0}^{\infty}3\left(-\frac{3}{4}\right)^n=3\sum_{n=0}^{\infty}\left(-\frac{3}{4}\right)^n=\frac{3}{1-(-\frac{3}{4})}=\frac{12}{7}$

e) $1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\cdots\frac{1}{4^8}=\frac{1-(\frac{1}{4})^9}{1-\frac{1}{4}}=\frac{\frac{262143}{262144}}{\frac{3}{4}}=\frac{87381}{65536}$

f) $1-\frac{1}{4}+\frac{1}{16}-\frac{1}{64}+\cdots\frac{1}{4^8}=\frac{1-(-\frac{1}{4})^9}{1-(-\frac{1}{4})}=\frac{\frac{262145}{262144}}{\frac{5}{4}}=\frac{52429}{65536}$

g) $1+\frac{3}{5}+\frac{9}{25}+\frac{27}{125}+\cdots\left(\frac{3}{5}\right)^8=\frac{1-\left(\frac{3}{5}\right)^9}{1-\frac{3}{5}}=\frac{\frac{1933442}{1953125}}{\frac{2}{5}}=\frac{966721}{390625}$

2. *Let $\angle ABC$ have measure α and let the points $A_1, A_2, A_3, \ldots$ be defined as follows:*

$A_1 = A$

A_2 *is the foot of the perpendicular from* A_1 *to BC,*

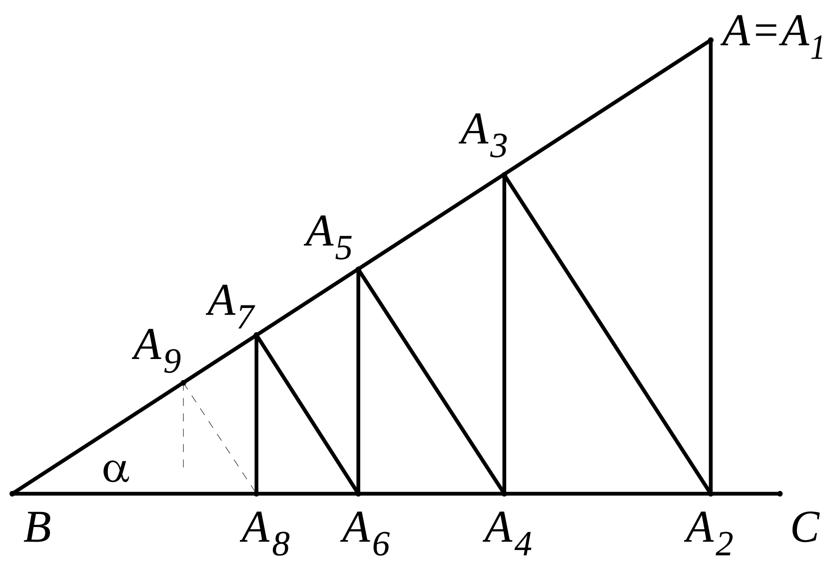
A_3 *is the foot of the perpendicular from* A_2 *to AB,*

A_4 *is the foot of the perpendicular from* A_3 *to BC,*

A_5 *is the foot of the perpendicular from* A_4 *to AB,*

…

Show that $A_1A_2 + A_2A_3 + A_3A_4 + \ldots = \frac{A_1A_2}{1-\cos\alpha}$.



First let's note that $\cos(\alpha) = \dfrac{BA_2}{AB}$ and

$$\Delta ABA_2 \sim \Delta A_1A_2A_3 \sim \Delta A_2A_3A_4 \sim \cdots \sim \Delta A_iA_{i+1}A_{i+2} \sim \cdots$$

since the "obvious" lines above are parallel. Note that this means that

$$\cos(\alpha) = \frac{BA_2}{AB} = \frac{A_2A_3}{A_1A_2} = \frac{A_3A_4}{A_2A_3} = \cdots = \frac{A_{i+1}A_{i+2}}{A_iA_{i+1}} = \cdots$$

$$\begin{aligned}
A_2A_3 &= A_1A_2\cos(\alpha) \\
A_3A_4 &= A_2A_3\cos(\alpha) = A_1A_2\cos^2(\alpha) \\
A_4A_5 &= A_3A_4\cos(\alpha) = A_1A_2\cos^3(\alpha) \\
A_5A_6 &= A_4A_5\cos(\alpha) = A_1A_2\cos^3(\alpha) \\
&\vdots \\
A_kA_{k+1} &= A_{k-1}A_k\cos(\alpha) = A_1A_2\cos^{k-1}(\alpha) \\
&\vdots
\end{aligned}$$

Therefore,

$$\begin{aligned}
A_1A_2 + A_2A_3 + A_3A_4 + \ldots &= A_1A_2 + A_1A_2\cos(\alpha) + A_1A_2\cos^2(\alpha) + \cdots + A_1A_2\cos^n(\alpha) + \cdots \\
&= A_1A_2\left(1 + \cos(\alpha) + \cos^2(\alpha) + \cdots + \cos^n(\alpha) + \cdots\right) \\
&= \frac{A_1A_2}{1-\cos\alpha} \text{ if } -1 < \cos\alpha < 1 \text{ if } 0 < \alpha < \frac{\pi}{2}
\end{aligned}$$

3. *The rightmost half of the unit interval* [0,1] *is removed. Next, the rightmost third of the remaining interval is removed. At the third stage the rightmost quarter of the remaining interval is removed and so on. Compute the sum of the lengths of all the extracted intervals.*

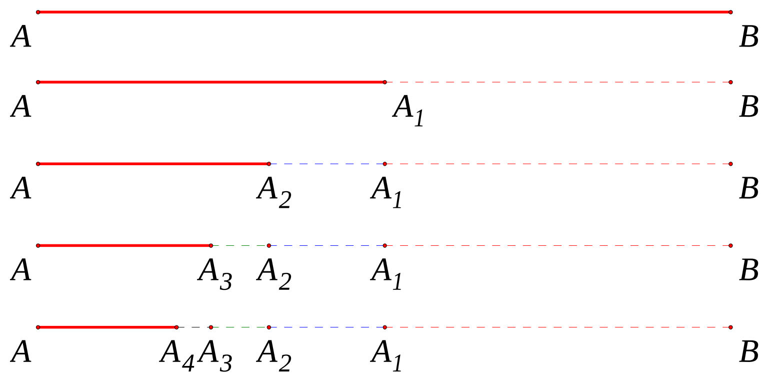
Stage 0:

Stage 1:

Stage 2:

Stage 3:

Stage 4:



We are told that

$$A_1B = AA_1 = \frac{1}{2}$$

$$A_1A_2 = \frac{1}{3}AA_1$$

$$A_2A_3 = \frac{1}{4}AA_2$$

$$A_3A_4 = \frac{1}{5}AA_3$$

$$\vdots$$

$$A_nA_{n+1} = \frac{1}{n+2}AA_n$$

Let's find each of these in terms of the length of AB, which is 1.

$$A_1B = AA_1 = \frac{1}{2}$$

$$A_1A_2 = \frac{1}{3}AA_1 = \frac{1}{2\cdot 3}$$

$$A_2A_3 = \frac{1}{4}AA_2 = \frac{1}{4}\left(AA_1 - A_1A_2\right) = \frac{1}{4}\left(\frac{1}{2} - \frac{1}{2}\frac{1}{3}\right) = \frac{1}{3\cdot 4}$$

$$A_3A_4 = \frac{1}{5}AA_3 = \frac{1}{5}\left(AA_2 - A_2A_3\right) = \frac{1}{5}\left(\frac{1}{3} - \frac{1}{4}\frac{1}{3}\right) = \frac{1}{4\cdot 5}$$

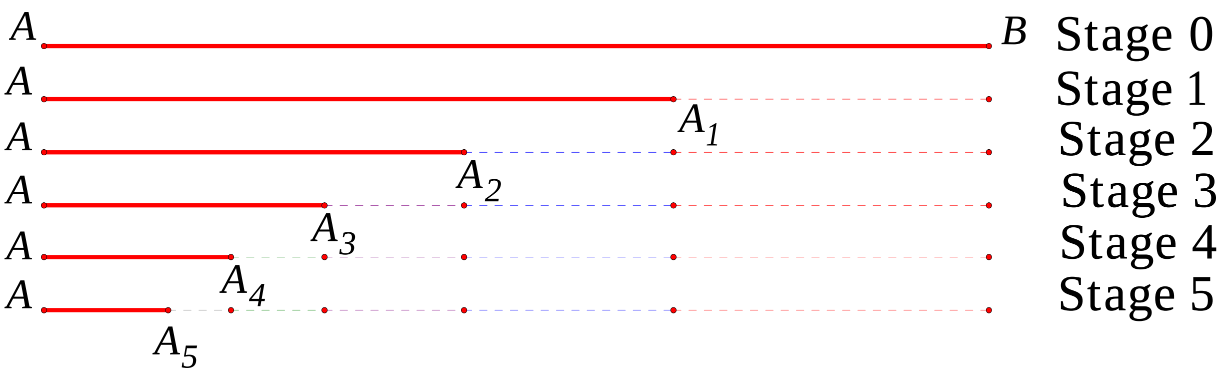
$$\vdots$$

$$A_{n-1}A_n = \frac{1}{n+1}AA_{n-1} = \frac{1}{n+1}\left(AA_{n-1} - A_{n-1}A_n\right) = \frac{1}{n+1}\left(\frac{1}{n-1} - \frac{1}{n}\frac{1}{n-1}\right) = \frac{1}{n\cdot(n+1)}$$

So, if we add up the lengths that are removed we get R:

$$\begin{aligned} R &= BA_1 + A_1A_2 + A_2A_3 + \cdots + A_{n-1}A_n + \cdots \\ &= \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots + \frac{1}{n(n+1)} + \cdots \\ &= \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1 \end{aligned}$$

4. *The rightmost third of the unit interval* $[0,1]$ *is removed. Next, the rightmost third of the remaining interval is removed. At the third stage the rightmost third of the remaining interval is removed and so on. Compute the sum of the lengths of all the extracted intervals.*



We are told that

$$\begin{aligned} A_1B &= \frac{1}{3}AB = \frac{1}{3} \\ A_1A_2 &= \frac{1}{3}AA_1 \\ A_2A_3 &= \frac{1}{3}A_1A_2 \\ &\vdots \\ A_nA_{n+1} &= \frac{1}{3}A_{n-1}A_n \end{aligned}$$

Let's find each of these in terms of the length of the unit interval, 1.

$$A_1B = \frac{1}{3} \Rightarrow AA_1 = \frac{2}{3}$$

$$A_1A_2 = \frac{1}{3}AA_1 \Rightarrow AA_2 = AA_1 - \frac{1}{3}AA_1 = \frac{2}{3}AA_1 = \left(\frac{2}{3}\right)^2$$

$$A_2A_3 = \frac{1}{3}AA_2 \Rightarrow AA_3 = AA_2 - \frac{1}{3}AA_2 = \frac{2}{3}AA_2 = \left(\frac{2}{3}\right)^3$$

$$\vdots$$

$$A_nA_{n+1} = \frac{1}{3}AA_n \Rightarrow AA_{n+1} = AA_n - \frac{1}{3}AA_n = \frac{2}{3}AA_n = \left(\frac{2}{3}\right)^{n+1}$$

The lengths that are removed are:

$$BA_1 + A_1A_2 + A_2A_3 + \cdots + A_{n-1}A_n = 1 - AA_n$$

and so the total length removed is:

$$\begin{aligned} R &= \lim_{n\to\infty}\left(1 - AA_n\right) \\ &= \lim_{n\to\infty}\left(1 - \left(\frac{2}{3}\right)^n\right) \\ &= 1 \end{aligned}$$