MATH 6101 090 Solutions Assignment 4

1. *Use the method of infinite series to solve the differential equation* $y' = 1 + x + y$.

Let $y = \sum_{n=0}^{\infty} a_n x^n$, then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$. Our equation now becomes:

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = 1 + x + \sum_{n=0}^{\infty} a_n x^n$$

$$a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots = (1 + a_0) + (1 + a_1)x + a_2 x^2 + a_3 x^3 + \cdots$$

This gives the following set of equations:

$$a_1 = 1 + a_0$$

$$2a_2 = 1 + a_1 \Rightarrow \quad 2a_2 = 2 + a_0 \Rightarrow \quad a_2 = 1 + \frac{1}{2}a_0 = \frac{2 + a_0}{2}$$

$$3a_3 = a_2 \quad \Rightarrow \quad a_3 = \frac{1}{3}a_2 = \frac{1}{3} + \frac{1}{6}a_0 = \frac{2 + a_0}{2 \cdot 3}$$

$$4a_4 = a_3 \quad \Rightarrow \quad a_4 = \frac{1}{4}a_3 = \frac{1}{12} + \frac{1}{24}a_0 = \frac{2 + a_0}{2 \cdot 3 \cdot 4}$$

This gives us that the solution is:

$$y = a_0 + (1 + a_0)x + \sum_{n=2}^{\infty} \frac{2 + a_0}{n!} x^n$$

[Symbolically, we have $y = -2 - x + a_o e^x$]

2. *Use the method of infinite series to solve the differential equation* $xy' = 1 + y$.

Let $y = \sum_{n=0}^{\infty} a_n x^n$, then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$. Our equation now becomes:

$$x \sum_{n=1}^{\infty} n a_n x^{n-1} = 1 + \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=1}^{\infty} n a_n x^n = 1 + \sum_{n=0}^{\infty} a_n x^n$$

$$a_1 x + 2a_2 x^2 + 3a_3 x^3 + 4a_4 x^4 + \cdots = (1 + a_0) + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

This gives the following set of equations:

$$0 = 1 + a_0 \quad \Rightarrow \quad a_0 = -1$$

$$a_1 = a_1$$

$$2a_2 = a_2 \Rightarrow \quad a_2 = 0$$

$$3a_3 = a_3 \quad \Rightarrow \quad a_3 = 0$$

$$4a_4 = a_4 \quad \Rightarrow \quad a_4 = 0$$

This gives us that the solution is:

$$y = -1 + a_1 x$$

3. *Use the method of infinite series to solve the differential equation* $y'' = xy$.

Let $y = \sum_{n=0}^{\infty} a_n x^n$, then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$. Our equation now becomes:

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = x \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1}$$

$$2a_2 + 3\cdot 2a_3 x + 4\cdot 3a_4 x^2 + 5\cdot 4a_5 x^3 + 6\cdot 5a_6 x^4 + \cdots = a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + \cdots$$

This gives the following set of equations:

$$2a_2 = 0 \quad \Rightarrow \quad a_2 = 0$$

$$6a_3 = a_0 \quad \Rightarrow \quad a_3 = \frac{a_0}{2\cdot 3}$$

$$4\cdot 3a_4 = a_1 \Rightarrow \quad a_4 = \frac{a_1}{3\cdot 4}$$

$$5\cdot 4a_5 = a_2 \quad \Rightarrow \quad a_5 = \frac{1}{4\cdot 5} a_2 = 0$$

$$6\cdot 5a_6 = a_3 \quad \Rightarrow \quad a_6 = \frac{a_3}{5\cdot 6} = \frac{a_0}{2\cdot 3\cdot 5\cdot 6}$$

$$7\cdot 6a_7 = a_4 \quad \Rightarrow \quad a_7 = \frac{a_4}{6\cdot 7} = \frac{a_1}{3\cdot 4\cdot 6\cdot 7}$$

This gives us that the solution is:

$$y = a_0 + a_1 x + \frac{1}{6} a_0 x^3 + \frac{1}{12} a_1 x^4 + \frac{1}{180} a_0 x^6 + \frac{1}{504} a_1 x^7 + \cdots$$

$$= a_0 \left(1 + \frac{1}{6} x^3 + \frac{1}{180} x^6 + \cdots\right) + a_1 \left(x + \frac{1}{12} x^4 + \frac{1}{504} x^7 + \cdots\right)$$

4. *Find the first three terms of the infinite expansion of y in terms of x for the equation* $y - 2xy + x^2 y = 1$

Set $F(x, y) = y - 2xy + x^2 y - 1 = 0$.

Stage 0: Replace x by $t_0 = \sum_{n=0}^{\infty} a_n x^n$ and extract the 0th level equation, i.e., the constants:

$$F_0(x, t_0) = t_0 - 2x t_0 + x^2 t_0 - 1 = 0$$

$$a_0 - 1 = 0$$

$$a_0 = 1$$

Stage 1: Substitute $t_0 = 1 + t_1$ into $F_0(x, t_0)$ to get $F_1(x, t_1)$ and extract the linear terms.

$F_1(x,t_1) = (1+t_1) - 2x(1+t_1) + x^2(1+t_1) - 1 = 0$

$F_1(x,t_1) = t_1 - 2xt_1 + x^2t_1 - 2x + x^2 = 0$

$a_1 - 2 = 0$

$a_1 = 2$

Stage 2: Substitute $t_1 = 2x + t_2$ into $F_1(x,t_1)$ to get $F_2(x,t_2)$ and extract the quadratic terms.

$F_2(x,t_2) = (2x+t_2) - 2x(2x+t_2) + x^2(2x+t_2) - 2x + x^2 = 0$

$F_2(x,t_2) = t_2 - 2xt_2 + x^2t_2 - 3x^2 + 2x^3 = 0$

$a_2 - 3 = 0$

$a_2 = 3$

So the first three terms are $y = 1 + 2x + 3x^2 + \cdots$.

Note that if we were to solve this equation for y, then we get:

$y - 2xy + x^2y = 1$

$$y = \frac{1}{x^2 - 2x + 1} = \frac{1}{(x-1)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=1}^{\infty} nx^{n-1}$$

5. *Find the first three terms of the infinite expansion of y in terms of x for the equation* $y^2 - y = x$

Set $F(x,y) = y^2 - y - x = 0$.

Stage 0: Replace x by $t_0 = \sum_{n=0}^{\infty} a_n x^n$ and extract the 0th level equation, i.e., the constants:

$F_0(x,t_0) = t_0{}^2 - t_0 - x = 0$

$a_0^2 - a_0 = 0$

$a_0 = 0,1$

Solution 1: Let a_0=0.

Stage 1: Substitute $t_0 = t_1$ into $F_0(x,t_0)$ to get $F_1(x,t_1)$ and extract the linear terms.

$F_1(x,t_1) = t_1{}^2 - t_1 - x = 0$

$-a_1 - 1 = 0$

$a_1 = -1$

Stage 2: Substitute $t_1 = -x + t_2$ into $F_1(x,t_1)$ to get $F_2(x,t_2)$ and extract the quadratic terms.

$F_2(x,t_2) = (-x+t_2)^2 - (-x+t_2) - x = 0$

$F_2(x,t_2) = -t_2 - 2xt_2 + t_2{}^2 + x^2 = 0$

$-a_2 + 1 = 0$

$a_2 = 1$

Stage 3: Substitute $t_2 = x^2 + t_3$ into $F_2(x,t_2)$ to get $F_3(x,t_3)$ and extract the degree three coefficients.

$F_3(x,t_3) = -(x^2+t_3) - 2x(x^2+t_3) + (x^2+t_3)^2 + x^2 = 0$

$F_3(x,t_2) = -t_3 + t_3^2 - 2xt_3 + 2x^2t_3 - 2x^3 + x^4 = 0$

$-a_3 - 2 = 0$

$a_3 = -2$

So the first three non-zero terms are $y = -x + x^2 - 2x^3 + \cdots$.

Solution 2: Now, let a_0=1.

Stage 1: Substitute $t_0 = 1 + t_1$ into $F_0(x,t_0)$ to get $F_1(x,t_1)$ and extract the linear terms.

$F_1(x,t_1) = (1+t_1)^2 - (1+t_1) - x = 0$

$F_1(x,t_1) = t_1 + t_1^2 - x = 0$

$a_1 - 1 = 0$

$a_1 = 1$

Stage 2: Substitute $t_1 = x + t_2$ into $F_1(x,t_1)$ to get $F_2(x,t_2)$ and extract the quadratic terms.

$F_2(x,t_2) = (x+t_2) + (x+t_2)^2 - x = 0$

$F_2(x,t_2) = t_2 + 2xt_2 + t_2{}^2 + x^2 = 0$

$a_2 + 1 = 0$

$a_2 = -1$

Stage 3: Substitute $t_2 = -x^2 + t_3$ into $F_2(x,t_2)$ to get $F_3(x,t_3)$ and extract the degree three coefficients.

$F_3(x,t_3) = (-x^2+t_3) + 2x(-x^2+t_3) + (-x^2+t_3)^2 + x^2 = 0$

$F_2(x,t_2) = t_3 + t_3^2 + 2xt_3 - 2x^2t_3 - 2x^3 + x^4 = 0$

$a_3 - 2 = 0$

$a_3 = 2$

So the first four non-zero terms are $y = 1 + x - x^2 + 2x^3 + \cdots$.

Note that if we were to solve this equation for y, then we get:

$$
\begin{aligned}
y^2 - y - x &= 0 \\
y &= \frac{1 \pm \sqrt{1+4x}}{2} \\
y_1 &= \frac{1 + \sqrt{1+4x}}{2} \\
&= 1 + x - x^2 + 2x^3 - 5x^4 + 14x^5 - 42x^6 + 132x^7 - 429x^8 + 1430x^9 + \cdots \\
y_2 &= \frac{1 - \sqrt{1+4x}}{2} \\
&= -x + x^2 - 2x^3 + 5x^4 - 14x^5 + 42x^6 - 132x^7 + 429x^8 - 1430x^9 + \cdots
\end{aligned}
$$