MATH 6101 090 Solutions Assignment 5

1. *a) Show that*

$$\frac{x^3}{12}-\frac{\pi^2 x}{12}=-\sin x+\frac{\sin 2x}{2^3}-\frac{\sin 3x}{3^3}+\frac{\sin 4x}{4^3}-\cdots \quad \textit{for } -\pi<x<\pi$$

From Proposition 5.1.4 we have

$$-\frac{x^2}{4}+\frac{\pi^2}{12}=\cos x-\frac{1}{4}\cos 2x+\frac{1}{9}\cos 3x-\frac{1}{16}\cos 4x+\frac{1}{25}\cos 5x-\cdots$$

Integrating we get:

$$-\frac{x^3}{12}+\frac{x\pi^2}{12}+C=\sin x-\frac{1}{8}\sin 2x+\frac{1}{27}\sin 3x-\frac{1}{64}\sin 4x+\cdots$$

Since sin(0) = 0, we get:

$$\frac{x^3}{12}-\frac{x\pi^2}{12}=-\sin x+\frac{\sin 2x}{2^3}-\frac{\sin 3x}{3^3}+\frac{\sin 4x}{4^3}-\cdots$$

b) Show that

$$\frac{x^4}{48}-\frac{\pi^2 x^2}{24}+\frac{7\pi^4}{720}=\cos x-\frac{\cos 2x}{2^4}+\frac{\cos 3x}{3^4}-\frac{\cos 4x}{4^4}+\cdots \quad \textit{for } -\pi<x<\pi$$

From above

$$\frac{x^3}{12}-\frac{x\pi^2}{12}=-\sin x+\frac{\sin 2x}{2^3}-\frac{\sin 3x}{3^3}+\frac{\sin 4x}{4^3}-\cdots$$

Integrating we get:

$$\frac{x^4}{48}-\frac{\pi^2 x^2}{24}+C=\cos x-\frac{\cos 2x}{2^4}+\frac{\cos 3x}{3^4}-\frac{\cos 4x}{4^4}+\cdots$$

Since cos(0) = 1, we get:

$$C=1-\frac{1}{2^4}+\frac{1}{3^4}-\frac{1}{4^4}+\cdots$$

Likewise substituting $x=\pi/2$, we get:

$$\frac{\pi^4}{2^4\times 48}-\frac{\pi^4}{2^2\times 24}+C=\cos(\pi/2)-\frac{\cos\pi}{2^4}+\frac{\cos 3\pi/2}{3^4}-\frac{\cos 2\pi}{4^4}+\cdots$$

$$C-\frac{7\pi^4}{768}=\frac{1}{2^4}-\frac{1}{4^4}+\frac{1}{6^4}-\frac{1}{8^4}+\cdots=\frac{1}{2^4}\left(1-\frac{1}{2^4}+\frac{1}{3^4}-\frac{1}{4^4}+\cdots\right)$$

$$C-\frac{7\pi^4}{768}=\frac{C}{2^4}$$

$$C=\frac{7\pi^4}{720}$$

Thus, we get:

$$\frac{x^4}{48}-\frac{\pi^2 x^2}{24}+\frac{7\pi^4}{720}=\cos x-\frac{\cos 2x}{2^4}+\frac{\cos 3x}{3^4}-\frac{\cos 4x}{4^4}+\cdots$$

c) Show that

$$\frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots \; for \; -\pi < x < \pi$$

From above

$$\frac{x^4}{48} - \frac{\pi^2 x^2}{24} + \frac{7\pi^4}{720} = \cos x - \frac{\cos 2x}{2^4} + \frac{\cos 3x}{3^4} - \frac{\cos 4x}{4^4} + \cdots$$

Substituting $x = 0$, we get:

$$\frac{7\pi^4}{720} = 1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \cdots$$

$$= \left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots\right) - 2\left(\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \frac{1}{8^4} + \cdots\right)$$

$$= \left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots\right) - \frac{2}{2^4}\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots\right)$$

$$\frac{7\pi^4}{720} = \frac{14}{16}\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots\right)$$

$$\frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots$$

d) Show that

$$\frac{\pi^6}{945} = 1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \cdots \; for \; -\pi < x < \pi$$

We have to do a), b) and c) to get this.

$$\frac{x^4}{48} - \frac{\pi^2 x^2}{24} + \frac{7\pi^4}{720} = \cos x - \frac{\cos 2x}{2^4} + \frac{\cos 3x}{3^4} - \frac{\cos 4x}{4^4} + \cdots$$

Integrating we get:

$$\frac{x^5}{240} - \frac{\pi^2 x^3}{72} + \frac{7\pi^4 x}{720} + C = \sin x - \frac{\sin 2x}{2^5} + \frac{\sin 3x}{3^5} - \frac{\sin 4x}{4^5} + \cdots$$

Since sin(0) = 0, we get:

$$\frac{x^5}{240} - \frac{\pi^2 x^3}{72} + \frac{7\pi^4 x}{720} = \sin x - \frac{\sin 2x}{2^5} + \frac{\sin 3x}{3^5} - \frac{\sin 4x}{4^5} + \cdots$$

Integrating we get:

$$\frac{x^6}{1440} - \frac{\pi^2 x^4}{288} + \frac{7\pi^4 x^2}{1440} + C = -\cos x + \frac{\cos 2x}{2^6} - \frac{\cos 3x}{3^6} + \frac{\cos 4x}{4^6} + \cdots$$

Since cos(0) = 1, we get:

$$C = -1 + \frac{1}{2^6} - \frac{1}{3^6} + \frac{1}{4^6} - \cdots$$

Likewise substituting $x = \pi / 2$, we get:

$$\frac{\pi^6}{2^6\times 1440}-\frac{\pi^6}{2^4\times 288}+\frac{7\pi^6}{2^2\times 1440}+C=-\cos(\pi/2)+\frac{\cos\pi}{2^6}-\frac{\cos 3\pi/2}{3^6}+\frac{\cos 2\pi}{4^6}+\cdots$$

$$C+\frac{31\pi^6}{30720}=-\frac{1}{2^6}+\frac{1}{4^6}-\frac{1}{6^6}+\frac{1}{8^6}+\cdots=\frac{1}{2^6}\left(-1+\frac{1}{2^6}-\frac{1}{3^6}+\frac{1}{4^6}-\cdots\right)$$

$$C+\frac{31\pi^6}{30720}=\frac{C}{2^6}$$

$$C=-\frac{31\pi^6}{30240}$$

Thus, we get:

$$\frac{x^6}{1440}-\frac{\pi^2x^4}{288}+\frac{7\pi^4x^2}{1440}-\frac{31\pi^6}{30240}=-\cos x+\frac{\cos 2x}{2^6}-\frac{\cos 3x}{3^6}+\frac{\cos 4x}{4^6}+\cdots$$

Substituting $x = 0$, we get:

$$\frac{31\pi^6}{30240}=1-\frac{1}{2^6}+\frac{1}{3^6}-\frac{1}{4^6}+\cdots$$

$$=\left(1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\cdots\right)-2\left(\frac{1}{2^6}+\frac{1}{4^6}+\frac{1}{6^6}+\frac{1}{8^6}+\cdots\right)$$

$$=\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots\right)-\frac{2}{2^6}\left(1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\cdots\right)$$

$$\frac{31\pi^6}{30240}=\frac{62}{64}\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots\right)$$

$$\frac{\pi^6}{945}=1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\cdots$$

2. *Prove that* $\frac{\pi}{2\sqrt{2}}=1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{13}-\frac{1}{15}+\cdots$.

We have that

$$x=2\left(\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-\frac{1}{4}\sin 4x+\frac{1}{5}\sin 5x-\cdots\right)$$

$$\frac{\pi}{4}=2\left(\sin\frac{\pi}{4}-\frac{1}{2}\sin\frac{2\pi}{4}+\frac{1}{3}\sin\frac{3\pi}{4}-\frac{1}{4}\sin\frac{4\pi}{4}+\frac{1}{5}\sin\frac{5\pi}{4}-\frac{1}{6}\sin\frac{6\pi}{4}+\frac{1}{7}\sin\frac{7\pi}{4}-\cdots\right)$$

$$\frac{\pi}{4}=2\left(\frac{\sqrt{2}}{2}-\frac{1}{2}+\frac{1}{3}\frac{\sqrt{2}}{2}-\frac{1}{5}\frac{\sqrt{2}}{2}+\frac{1}{6}-\frac{1}{7}\frac{\sqrt{2}}{2}-\cdots\right)$$

$$\frac{\pi}{4}=\sqrt{2}\left(1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}-\cdots\right)+2\left(-\frac{1}{2}+\frac{1}{6}-\frac{1}{10}+\frac{1}{14}-\cdots\right)$$

$$\frac{\pi}{4}=\sqrt{2}\left(1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}-\cdots\right)-\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\right)$$

$$\frac{\pi}{4}=\sqrt{2}\left(1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}-\cdots\right)-\frac{\pi}{4}$$

$$\frac{\pi}{2}=\sqrt{2}\left(1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}-\cdots\right)$$

$$\frac{\pi}{2\sqrt{2}}=1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}\cdots$$

3. *Prove that* $\frac{4\pi^2}{27}=1+\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{8^2}+\frac{1}{10^2}+\frac{1}{11^2}+\cdots$.

   Here we will use

$$x^2=4\left(\frac{\pi^2}{12}-\cos x+\frac{\cos 2x}{2^2}-\frac{\cos 3x}{3^2}+\frac{\cos 4x}{4^2}-\frac{\cos 5x}{5^2}-\cdots\right)$$

   with an appropriate choice for x: $x=\frac{\pi}{3}$

$$\left(\frac{\pi}{3}\right)^2=4\left(\frac{\pi^2}{12}-\cos\frac{\pi}{3}+\frac{\cos(2\pi/3)}{2^2}-\frac{\cos(3\pi/3)}{3^2}+\frac{\cos(4\pi/3)}{4^2}-\frac{\cos(5\pi/3)}{5^2}+\frac{\cos(6\pi/3)}{6^2}-\cdots\right)$$

$$\frac{\pi^2}{9}=4\left(\frac{\pi^2}{12}-\frac{1}{2}-\frac{1/2}{2^2}+\frac{1}{3^2}-\frac{1/2}{4^2}-\frac{1/2}{5^2}+\frac{1}{6^2}-\cdots\right)$$

$$\frac{\pi^2}{9}=\frac{\pi^2}{3}-\frac{4}{2}\left(1+\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots\right)+4\left(\frac{1}{3^2}+\frac{1}{6^2}+\frac{1}{9^2}+\frac{1}{12^2}+\cdots\right)$$

$$\frac{\pi^2}{9}=\frac{\pi^2}{3}-2\left(1+\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots\right)+\frac{4}{3^2}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right)$$

$$\frac{\pi^2}{9}=\frac{\pi^2}{3}-2\left(1+\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots\right)+\frac{4}{3^2}\frac{\pi^2}{6}$$

$$\frac{1}{2}\left(\frac{\pi^2}{3}+\frac{2\pi^2}{27}-\frac{\pi^2}{9}\right)=1+\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots$$

$$\frac{4\pi^2}{27}=1+\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{8^2}+\frac{1}{10^2}+\frac{1}{11^2}+\cdots$$

4. *Prove that* $\frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots$.

   Use what Euler already had

$$\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\cdots$$

   and

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \cdots$$

Therefore,

$$\frac{\pi^2}{6} + \frac{\pi^2}{12} = \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots\right) + \left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \cdots\right)$$

$$\frac{\pi^2}{4} = 2 + \frac{2}{3^2} + \frac{2}{5^2} + \cdots$$

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

5. *Prove that* $\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots$.

From 4(a) we have.

$$\frac{x^3}{12} - \frac{\pi^2 x}{12} = -\sin x + \frac{\sin 2x}{2^3} - \frac{\sin 3x}{3^3} + \frac{\sin 4x}{4^3} - \cdots$$

Let $x = \frac{\pi}{2}$:

$$\frac{\pi^3}{96} - \frac{\pi^3}{24} = -1 + \frac{1}{3^3} - \frac{1}{5^3} + \cdots$$

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \cdots$$