MATH 6101
Fall 2008

Calculus from Archimedes to Fermat

A Request

Please define a *relative maximum*.

Please define a *relative minimum*.

How can you tell them apart?

The Derivative: A Chronology

1. Used *ad hoc* to solve particular problems
2. Discovered as a general concept
3. Explored and developed in applications to mathematics and physics
4. Defined rigorously

Curves and Tangents

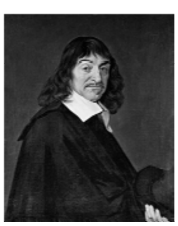
- Greeks (mainly known from work of Archimedes) had studied some curves
  - Circle
  - Conic sections (parabola, ellipse, hyperbola)
  - Spirals
  - Others defined as loci of points
- Muslim scholars studied a few more
- Many problems studied, especially finding their tangents and areas

Move to Medieval Europe

- Scholars of Europe began to study the classics of Greek mathematics as augmented by Muslim scholars
- 1591 – François Viète (Vieta) – *Isagoge in artem analyticam* introduced symbolic algebra (without an equal sign)

Algebra and Curves

In the 1630's Descartes and Fermat independently discovered/invented analytic geometry



Algebra and Curves

With this algebra there was an explosion of curves to study.

Greek method of synthetic geometry would not work.

New method required for finding tangents and areas

17-Sept-2008 MATH 6101 7

Algebra and Curves

- Tangents
- Areas
- Extrema – from the Greeks came isoperimetric problems – "Of all plane figures with the same perimeter, which one has the maximal area?"
- Fermat and Descartes had hopes for these being answered by symbolic algebra

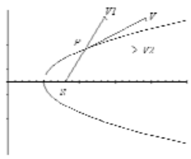
17-Sept-2008 MATH 6101 8

de Roberval's Method of Tangents

- Notion of *instantaneous motion*.
- A curve is sketched by a moving point.
- The tangent is the sum of vectors making up the motion.

17-Sept-2008 MATH 6101 9

de Roberval's Method of Tangents



- Parabola showing the motion vectors *V*1 and *V*2 at a point *P*.
- *V*1 is in the same direction as the line joining the focus of the parabola, *S*, and the point *P*.
- *V*2 is perpendicular to the directrix
- The tangent to the graph at point P is simply the vector sum *V* = *V*1 + *V*2

Found tangents to other curves including the ellipse and cycloid, but could not generalize

Fermat's Method of Derivatives

Fermat's Illustration:

Given a line, to divide it into two parts so that the product of the parts will be a maximum.

Let b = length of the line

a = length of the first part

$$a(b-a) = ab - a^2$$

Pappus of Alexandria – a problem which in general has two solutions will have only one solution in the case of a maximum

Fermat's Method

Suppose that there is a second solution. Then the first part of the line would be $a + e$ and the second would be $b - (a + e) = b - a - e$.

Multiply the two parts together:

$ba + be - a^2 - ae - ea - e^2 = ab - a^2 - 2ae + be - e^2$

By Pappus, there is only one solution so set these equal to one another:

$$ab - a^2 = ab - a^2 - 2ae + be - e^2$$

$$2ae + e^2 = be$$

Fermat's Method

$$ab - a^2 = ab - a^2 - 2ae + be - e^2$$
$$2ae + e^2 = be$$
$$2a + e = b$$

Now Fermat says "*suppress e*" and we get:

$$a = b/2$$

which is the point at which the maximum occurs.

Fermat's Method

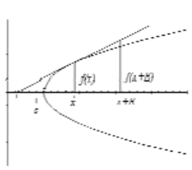
Note that Fermat did NOT:

- call e infinitely small
- say that e vanished;
- use a limit;
- explain why he could divide by e and then treat it as 0.

At this point he did not make the connection between this max-min method and finding tangents

Fermat's Method – Modern Notation

$$\lim_{e \to 0} \frac{f(x+e) - f(x)}{e}$$



- Finding tangents:
- Draws the tangent line at a point x and will consider a point a distance e away.
- From the figure, the following relationship exists:

$$\frac{s}{s+e} = \frac{f(x)}{f(x+e)}$$

Fermat's Method – Modern Notation

$$\frac{s}{s+e} = \frac{f(x)}{f(x+e)}$$

Solve for s

$$s = \frac{f(x)}{\left[f(x+e) - f(x)\right]/e}$$

The denominator is his differential

Slope = $f(x)/s$

Fermat's Method – Modern Notation

$$f(x) = x^4$$

$$s = \frac{f(x)}{\left[f(x+e) - f(x)\right]/e} = \frac{x^4}{\left[(x+e)^4 - x^4\right]/e}$$

$$s = \frac{x^4}{4x^3 + 6x^2e + 4xe^2 + e^3}$$

He sets $e = 0$.

$$s = \frac{x}{4} \quad \text{then} \quad f'(x) = \frac{f(x)}{s} = 4x^3$$

Fermat and Tangents

Using his method Fermat showed that the tangent to $y = x^n$ is always given by nx^{n-1}

Johann Hudde (1659) gave a general (verbal) form of the max-min problem in which he says (stated in modern notation):

Given a polynomial of the form $y = \sum_{k=0}^{n} a_k x^k$,

there is a maximum or minimum when

$$\sum_{k=1}^{n} k a_k x^{k-1} = 0.$$

Tangents

Descartes
Isaac Barrow
John Wallis
Rene Sluse
Christopher Huygens
All had methods of finding the tangent
By 1660 we had what is now known as Fermat's Theorem: *to find a maximum find where the tangent line has slope* 0.
Had no connection to the process of computing areas

17-Sept-2008 MATH 6101 19

Early Calculations of Area

- We say what Archimedes had done with the area between the parabola and a secant line.
- This was the only time that Archimedes used a geometric series preferring arithmetic series
- Areas of general curves needed symbolic algebra

17-Sept-2008 MATH 6101 20

Bonaventura Cavalieri (1598 – 1647)

- *Geometria indivisibilibus continuorum nova quadam ratione promota* (1635)
- Development of Archimedes' method of exhaustion incorporating Kepler's theory of infinitesimally small geometric quantities.
- Allowed him to find simply and rapidly area and volume of various geometric figures.

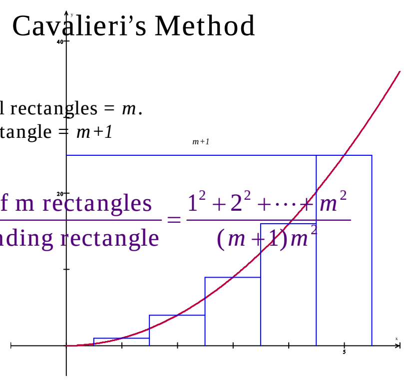
17-Sept-2008 MATH 6101 21

Cavalieri’s Method of Indivisibles

- A moving point sketches a curve
- He viewed the curve as the sum of its points, or "indivisibles“
- Likewise, the "indivisibles" that composed an area were an infinite number of lines
- Kepler had done so before him, but he was the first to use this in the computation of areas

Cavalieri’s Method

base = 1
height = x^2
Number of small rectangles = m.
base of large rectangle = $m+1$
height = m^2



$$\frac{\text{Total area of m rectangles}}{\text{Area of bounding rectangle}} = \frac{1^2 + 2^2 + \cdots + m^2}{(m+1)m^2}$$

Cavalieri’s Method

Cavalieri computed this ratio for a large number of values of m. He noticed

$$\frac{\text{Total area of m rectangles}}{\text{Area of bounding rectangle}} = \frac{1}{3} + \frac{1}{6m}$$

He noticed that as he let m grow larger, the term $1/6m$ had less influence on the outcome of the result.

Uses the concept of infinity to describe the ratios of the area, he derives expression for area underneath the parabola.

Cavalieri's Method

For at any distance x along the x-axis, the height of the parabola would be x^2. Therefore, the area of the rectangle enclosing the rectangular subdivisions at a point x was equal to $(x)(x^2)$ or x^3.

From his earlier result, the area underneath the parabola is equal to 1/3 the area of the bounding rectangle

$$\text{Area under } x^2 = \frac{1}{3}x^3$$

17-Sept-2008 MATH 6101 25

John Wallis

Wallis showed that the area function for the curve $y = kx^n$ is

$$A = \frac{1}{n+1}kx^{n+1}$$

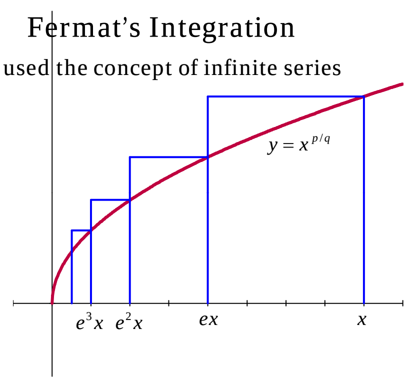
is true not only for positive integers but for negative and fractional exponents as well.

Also integrated polynomials

17-Sept-2008 MATH 6101 26

Fermat's Integration

- Fermat used the concept of infinite series



17-Sept-2008 MATH 6101 27

Fermat's Integration

- Choose $0 < e < 1$

$$(x - ex)x^{p/q} = x(1-e)x^{p/q} = (1-e)x^{p+q/q}$$

$$(ex - e^2x)(ex)^{p/q} = ex(1-e)(ex)^{p/q} = (1-e)e^{p+q/q}x^{p+q/q}$$

$$(e^2x - e^3x)x^{p/q} = e^2x(1-e)(e^2x)^{p/q} = (1-e)\left(e^2\right)^{p+q/q} x^{p+q/q}$$

- Adding these up, we get

$$(1-e)x^{p+q/q}\left(1 + e^{p+q/q} + \left(e^2\right)^{p+q/q} + \left(e^3\right)^{p+q/q} + \cdots\right)$$

17-Sept-2008 MATH 6101 28

Fermat's Integration

$$(1-e)x^{p+q/q}\left(1 + e^{p+q/q} + \left(e^2\right)^{p+q/q} + \left(e^3\right)^{p+q/q} + \cdots\right) =$$

$$= (1-e)x^{p+q/q}\frac{1}{1-e^{p+q/q}}$$

Substitute $e = E^q$

$$A = (1-e)x^{p+q/q}\frac{1}{1-e^{p+q/q}} = \frac{1-E^q}{1-E^{p+q}}x^{p+q/q}$$

17-Sept-2008 MATH 6101 29

Fermat's Integration

$$A = (1-e)x^{p+q/q}\frac{1}{1-e^{p+q/q}} = \frac{1-E^q}{1-E^{p+q}}x^{p+q/q}$$

$$A = \frac{(1-E)(1+E+E^2+\cdots+E^{q-1})}{(1-E)(1+E+E^2+\cdots+E^{p+q-1})}x^{p+q/q}$$

$$A = \frac{(1+E+E^2+\cdots+E^{q-1})}{(1+E+E^2+\cdots+E^{p+q-1})}x^{p+q/q}$$

17-Sept-2008 MATH 6101 30

Fermat's Integration

Let $E = 1$. Then

$$A = \frac{(1+1+1^2+\cdots+1^{q-1})}{(1+1+1^2+\cdots+1^{p+q-1})} x^{p+q/q} = \left(\frac{q}{p+q}\right) x^{p+q/q}$$