

# MATH 6101

## Fall 2008

### Infinite Series and Convergence



# Definition

Given any sequence  $\{a_n\}$  we associate a new sequence  $\{s_n\}$  of *partial sums*:

$$s_n = a_1 + a_2 + a_3 + a_4 + \dots + a_n$$

We define the ***series***  $\sum a_n$  to be the limit:

$$\sum a_n = \lim_{n \rightarrow \infty} s_n$$

If the sequence of partial sums converges, we say that the infinite series *converges*. Otherwise, we say that the series is *divergent*.

# Examples

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \quad |a| < 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = 2$$

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = 6$$

$$\sum_{n=1}^{\infty} \frac{n^3}{2^n} = 26$$

# Examples

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \quad |a| < 1$$

In this case we have seen that:

$$s_n = \sum_{k=0}^n a^k = 1 + a + a^2 + a^3 + \cdots + a^n = \frac{1 - a^{n+1}}{1 - a}$$

# Examples

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

In this case we have seen that:

$$\begin{aligned} S_n &= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \dots + \frac{1}{(n-1) \cdot n} \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \dots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &= 1 - \frac{1}{n} \end{aligned}$$

# Other Examples

Do these converge or diverge?

$$1 + 2 + 3 + 4 + 5 + \cdots + n + \cdots = \sum_{n=1}^{\infty} n$$

$$1 + 1 + 1 + 1 + 1 + \cdots = \sum_{n=1}^{\infty} 1$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n}$$

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

# First Series

$$\sum_{k=1}^{\infty} k = 1 + 2 + 3 + 4 + 5 + \cdots + k + \cdots$$

$$s_n = 1 + 2 + 3 + 4 + \cdots + n$$

$$s_n = \frac{n(n+1)}{2}$$

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{n(n+1)}{2} = +\infty$$

Thus, the limit of the sequence of partial sums does not exist as a real number, and the series diverges.

# Second Series

$$\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + 1 + \dots$$

$$s_n = \underbrace{1 + 1 + \dots + 1}_{n \text{ times}} = n$$

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} n = +\infty$$

Again, the sequence of partial sums does not exist as a real number, and the series diverges.

# Third Series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} + \dots = \sum_{n=1}^{\infty} \frac{1}{n}$$

This one is more difficult to see, but in 1350 Nicole Oresme proves the following:

$$\begin{aligned} 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}\right) + \dots \\ &= 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{> \frac{1}{4} + \frac{1}{4}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{> \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}} + \underbrace{\left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}\right)}_{> \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}} + \dots \\ &> 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots \end{aligned}$$

So this one does not add up to a finite number.

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Does this converge or diverge?

We know that  $2n^2 \geq n(n+1)$  so

$$\frac{2}{n(n+1)} \geq \frac{1}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \leq \sum_{n=1}^{\infty} \frac{2}{n(n+1)} = 2$$

Therefore, it does converges.

# Continued

We noted earlier that Euler proved in 1735 that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots = \frac{\pi^2}{6}$$

We also know more:

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{2^{2k-1} |B_{2k}| \pi^{2k}}{(2k)!}$$

where  $B_n$  is the  $n$ th *Bernoulli* number. Euler only went through the exponent 26.

# Bernoulli Numbers

The Bernoulli numbers  $B_n$  were discovered by Jakob Bernoulli in conjunction with computing the sums of powers:

$$\sum_{k=0}^{m-1} k^n = 0^n + 1^n + 2^n + 3^n + 4^n + \cdots + (m-1)^n$$

For example:

$$\sum_{k=0}^n k = \frac{1}{2}n^2 + \frac{1}{2}n$$

$$\sum_{k=0}^n k^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

# Bernoulli Numbers

$$\sum_{k=0}^n k^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$$

$$\sum_{k=0}^n k^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$$

$$\sum_{k=0}^n k^5 = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2$$

$$\sum_{k=0}^n k^6 = \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n$$

$$\sum_{k=0}^n k^7 = \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 - \frac{7}{24}n^4 + \frac{1}{12}n^2$$

$$\sum_{k=0}^n k^8 = \frac{1}{9}n^9 + \frac{1}{2}n^8 + \frac{2}{3}n^7 - \frac{7}{15}n^5 + \frac{2}{9}n^3 - \frac{1}{30}n$$

# Bernoulli Numbers

$$\sum_{k=0}^n k^9 = \frac{1}{10}n^{10} + \frac{1}{2}n^9 + \frac{3}{4}n^8 - \frac{7}{10}n^6 + \frac{1}{2}n^4 - \frac{1}{12}n^2$$

$$\sum_{k=0}^n k^{10} = \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 - n^7 + n^5 - \frac{1}{2}n^3 + \frac{5}{66}n$$

$$\sum_{k=0}^n k^{11} = \frac{1}{12}n^{12} + \frac{1}{2}n^{11} + \frac{11}{12}n^{10} - \frac{11}{8}n^8 + \frac{11}{6}n^6 - \frac{11}{8}n^4 + \frac{5}{12}n^2$$

$$\sum_{k=0}^n k^{12} = \frac{1}{13}n^{13} + \frac{1}{2}n^{12} + n^{11} - \frac{11}{6}n^9 + \frac{22}{7}n^7 - \frac{33}{10}n^5 + \frac{5}{3}n^4 - \frac{691}{2730}n$$

# Bernoulli Numbers

Bernoulli then states:

$$\begin{aligned} \sum_{k=0}^n k^p &= \frac{1}{p+1} n^{p+1} + \frac{1}{2} n^p + \frac{p}{2} A n^{p-1} + \frac{p(p-1)(p-2)}{2 \cdot 3 \cdot 4} B n^{p-3} \\ &+ \frac{p(p-1)(p-2)(p-3)(p-4)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} C n^{p-5} \\ &+ \frac{p(p-1)(p-2)(p-3)(p-4)(p-5)(p-6)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} D n^{p-7} + \dots \end{aligned}$$

where

$$A = \frac{1}{6}, \quad B = -\frac{1}{30}, \quad C = \frac{1}{42}, \quad D = -\frac{1}{30}$$

# Bernoulli Numbers

The  $p$ th Bernoulli number is the coefficient of  $n$  in the polynomial describing  $\sum k^p$ .

Other techniques for generating the Bernoulli numbers come from

$$\sum_{k=0}^{n-1} k^p = \frac{1}{p+1} \sum_{k=0}^p \binom{p+1}{k} B_k n^{p+1-k}$$

or

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$$

# Bernoulli Numbers

$n$	$B_n$	$n$	$B_n$
0	1	12	$-691/2730$
1	$-1/2$	14	$7/6$
2	$1/6$	16	$-3617/510$
4	$-1/30$	18	$43867/798$
6	$1/42$	20	$-174611/330$
8	$-1/30$	22	$854513/138$
10	$5/66$	24	$-236364091/2730$

$$B_{2n+1} = 0, \quad n > 0$$

# Series

We will be able to show later that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if  $p > 1$ . The easier proof requires a little calculus.



*"The next part of this recipe will involve some calculus."*

# Absolute Convergence

If the terms  $a_n$  of an infinite series  $\sum a_n$  are all nonnegative, then the partial sums  $\{s_n\}$  form a non-decreasing sequence.

Therefore,  $\sum a_n$  either converges or diverges to  $\infty$ .

$\sum |a_n|$  is non-decreasing for any sequence.

The series  $\sum a_n$  is said to *converge absolutely* if  $\sum |a_n|$  converges.

# Conditional Convergence

A series *converges conditionally*, if it converges, but not absolutely.

- Does the series  $\sum (-1)^n$  converge absolutely, conditionally, or not at all?
- Does the series  $\sum (1/2)^n$  converge absolutely, conditionally, or not at all?
- Does the series  $\sum (-1)^{n+1}/n$  converge absolutely, conditionally, or not at all (this series is called alternating harmonic series)?

# Order of Summation

## Theorem:

(i) *Let  $\sum a_n$  be an absolutely convergent series. Then any rearrangement of terms in that series results in a new series that is also absolutely convergent to the same limit.*

(ii) *Let  $\sum a_n$  be a conditionally convergent series. Then, for any real number  $c$  there is a rearrangement of the series such that the new resulting series will converge to  $c$ .*

(To be proven later)

# Algebra of Series

Let  $\sum a_n$  and  $\sum b_n$  be two absolutely convergent series.

Then

- (i) The sum of the two series is again absolutely convergent.  $\sum(a_n + b_n) = \sum a_n + \sum b_n$
- (ii) The difference of the two series is again absolutely convergent.  $\sum(a_n - b_n) = \sum a_n - \sum b_n$
- (iii) The product of the two series is again absolutely convergent. Its limit is the product of the limit of the two series.

# Algebra of Series

The Cauchy product of two series  $\sum a_n$  and  $\sum b_n$  is defined as follows. The Cauchy product is

$$\left( \sum_{n=m}^{\infty} a_n \right) \cdot \left( \sum_{n=m}^{\infty} b_n \right) = \left( \sum_{n=m}^{\infty} c_n \right) \text{ where } c_n = \sum_{k=0}^n a_k b_{n-k}$$

# *n*th Term Test

**Theorem:** *If  $\sum a_n$  converges then  $\{a_n\} \rightarrow 0$ .*

Metaproof: If  $\sum a_n$  converges, then the sequence of partial sums converges  $\{s_n\} \rightarrow L$ . Note that the sequence  $\{s_{n-1}\}$  also converges to  $L$ . Thus,

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (s_n - s_{n-1}) = \lim_{n \rightarrow \infty} s_n - \lim_{n \rightarrow \infty} s_{n-1} = L - L = 0.$$

**Corollary:** *If  $|a| \geq 1$  then  $\sum a^n$  diverges.*

**nth Term Test:** *If  $\lim a_n \neq 0$ , then  $\sum a_n$  diverges.*

# Comparison Test

**Theorem:** *If  $\sum a_n$  and  $\sum b_n$  are series so that*  
$$0 \leq a_n \leq b_n.$$

*Then*

*if  $\sum b_n$  converges so does  $\sum a_n$ ;*

*if  $\sum a_n$  diverges so does  $\sum b_n$ .*

# Comparison Test

Proof: Set

$$d_n = a_0 + a_1 + a_2 + \dots + a_n$$

$$e_n = b_0 + b_1 + b_2 + \dots + b_n$$

$\{d_n\}$  and  $\{e_n\}$  are increasing sequences.

$$0 \leq d_n \leq e_n$$

# Comparison Test

Each converges or diverges depending on whether it is bounded or not.

$\sum b_n$  converges  $\implies \{e_n\}$  converges  $\implies \{e_n\}$   
bounded  $\implies \{d_n\}$  bounded  $\implies \{d_n\}$   
converges  $\implies \sum a_n$  converges

$\sum a_n$  diverges  $\implies \{d_n\}$  diverges  $\implies \{d_n\}$   
unbounded  $\implies \{e_n\}$  unbounded  $\implies \{e_n\}$   
diverges  $\implies \sum b_n$  diverges

# Examples

$\sum_{n=1}^{\infty} \frac{1}{n2^n}$  converges or diverges?

$\frac{1}{n2^n} < \frac{1}{2^n}$  and we know that the latter converges

# Limit Comparison Test

## Theorem:

*Let  $\sum a_n$  and  $\sum b_n$  be two series. Suppose also*

$$r = \lim |a_n/b_n| \text{ exists and } 0 < r < +\infty.$$

*Then  $\sum a_n$  converges absolutely if and only if  $\sum b_n$  converges absolutely.*

# Limit Comparison Test

Proof:  $r = \lim |a_n/b_n|$  and  $r$  is a positive real number. There are constants  $c$  and  $C$ ,

$$0 < c < C < +\infty$$

so that for some  $N > 1$  if  $n > N$

$$c < |a_n/b_n| < C.$$

Assume  $\sum a_n$  converges absolutely. For  $n > N$ ,  $c|b_n| < |a_n|$ . Therefore,  $\sum b_n$  converges absolutely by the Comparison Test.

# Limit Comparison Test

Assume that  $\sum b_n$  converges absolutely. For  $n > N$ ,  
 $|a_n| < C|b_n|$ .

$C \sum b_n$  converges absolutely.  $\sum a_n$  converges  
absolutely by Comparison Test.

# Cauchy Condensation Test

## Theorem:

*Suppose  $\{a_n\}$  is a decreasing sequence of positive terms. Then the series  $\sum a_n$  converges if and only if the series  $\sum 2^k a_{2^k}$  converges.*

# $p$ -series Test

## Corollary:

*For a positive number  $p$ ,  $\sum 1/n^p$  converges if and only if  $p > 1$ .*

# $p$ -series Test

Proof:

If  $p < 0$  then the sequence  $\{1/n^p\}$  diverges to infinity. Hence, the series diverges by the  $n$ th Term Test.

If  $p > 0$  then consider the series

$$\sum 2^n a_{2^n} = \sum 2^n / (2^n)^p = \sum (2^{1-p})^n.$$

By the geometric series,

- if  $0 < p \leq 1$ ,  $2^{1-p} \geq 1$ , so right-hand series diverges;
- if  $p > 1$  then  $2^{1-p} < 1$ , so right-hand series converges.

Now the result follows from the Cauchy Condensation Test.

# Root Test

## Theorem:

Let  $\sum a_n$  be a series and let

$$\alpha = \limsup |a_n|^{1/n}.$$

The series  $\sum a_n$

- i. converges absolutely if  $\alpha < 1$ ,
- ii. diverges if  $\alpha > 1$ .
- iii. Otherwise  $\alpha = 1$  and the test gives no information.

# Root Test

Proof:

i) Suppose that  $\alpha < 1$ . Then choose an  $\varepsilon > 0$  so that  $\alpha + \varepsilon < 1$ . By definition of  $\limsup$  there exists  $N$  so that  $\alpha - \varepsilon < \text{lub}\{|a_n|^{1/n} \mid n > N, \alpha + \varepsilon\}$ . In particular,  $|a_n|^{1/n} < \alpha + \varepsilon$  for  $n > N$ , so  $|a_n| < (\alpha + \varepsilon)^n$  for  $n > N$ . Since  $0 < \alpha + \varepsilon < 1$ , the geometric series  $\sum(\alpha + \varepsilon)^n$  converges. By the Comparison Test  $\sum|a_n|$  converges. This means that  $\sum a_n$  also converges.

# Root Test

Proof:

- i) If  $\alpha > 1$ , then there is a subsequence of  $|a_n|^{1/n}$  that has limit  $\alpha > 1$ . That means that  $|a_n| > 1$  for infinitely many  $n$ . The sequence  $\{a_n\}$  cannot converge to 0, so  $\sum a_n$  cannot converge.
- ii) For the series  $\sum 1/n$  and  $\sum 1/n^2$ ,  $\alpha = 1$ . The harmonic series diverges and the other converges, so  $\alpha = 1$  can not guarantee either convergence or divergence of the series.

# Ratio Test

## Theorem:

The series  $\sum a_n$

i. converges absolutely if  $\limsup |a_{n+1}/a_n| < 1$ ,

ii. diverges if  $\liminf |a_{n+1}/a_n| > 1$ .

iii. Otherwise

$\liminf |a_{n+1}/a_n| \leq 1 \leq \limsup |a_{n+1}/a_n|$   
and the test gives no information.

# Alternating Series Test

## Theorem:

If  $a_1 \geq a_2 \geq \dots \geq a_n \geq \dots \geq 0$  and  $\{a_n\}$  converges to zero, then the alternating series  $\sum (-1)^n a_n$  converges.

# Problems

$$\lim_{n \rightarrow \infty} \sqrt[n]{a^n + b^n}$$

$$\lim_{n \rightarrow \infty} \left( n - \sqrt{n+a} \sqrt{n+b} \right)$$

$$\lim_{n \rightarrow \infty} \frac{a^n - b^n}{a^n + b^n}$$

$$\lim_{n \rightarrow \infty} \frac{2^{n^2}}{n!}$$

# Problems

$$\sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^2}$$

# Problems

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n^2 - 1}}$$

# Problems

$$\sum_{n=1}^{\infty} (-1)^n \frac{\log n}{n}$$

# Problems

$$\sum_{n=1}^{\infty} \frac{n^2}{n!}$$

# Problems

$$\sum_{n=1}^{\infty} \frac{\log n}{n}$$

# Problems

$$\sum_{n=2}^{\infty} \frac{1}{\log n}$$

# Problems

$$\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$$

# Problems

$$\sum_{n=0}^{\infty} \frac{n^2}{n^3 + 1}$$

# Problems

$$\sum_{n=2}^{\infty} \frac{1}{n^2 \log n}$$

# Problems

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$$