MATH 6101
Fall 2008

Series and a Famous Unsolved Problem

Problems

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)}=\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}+\cdots$$

$$\frac{1}{(2n-1)(2n+1)}=\frac{1}{2}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)$$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)}&=\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}+\cdots\\&=\frac{1}{2}\left[\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\cdots\right]\\&=\frac{1}{2}\end{aligned}$$

Problems

$$\sum_{n=1}^{\infty}\frac{1}{n(n+4)} \qquad \frac{1}{n(n+4)}=\frac{1}{4}\left(\frac{1}{n}-\frac{1}{n+4}\right)$$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n(n+4)}&=\frac{1}{1\cdot 5}+\frac{1}{2\cdot 6}+\frac{1}{3\cdot 7}+\frac{1}{4\cdot 8}+\frac{1}{5\cdot 9}+\cdots\\&=\frac{1}{4}\left[\left(1-\frac{1}{5}\right)+\left(\frac{1}{2}-\frac{1}{6}\right)+\left(\frac{1}{3}-\frac{1}{7}\right)+\left(\frac{1}{4}-\frac{1}{8}\right)+\left(\frac{1}{5}-\frac{1}{9}\right)+\cdots\right]\\&=\frac{1}{4}\left[1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right]=\frac{25}{48}\end{aligned}$$

Problems

$$\sum_{n=1}^{\infty}\frac{n}{(n+1)!} \quad a_n=\frac{n}{(n+1)!},\, a_{n+1}=\frac{n+1}{(n+2)!}$$

$$q=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{n+1}{(n+2)!}\cdot\frac{(n+1)!}{n}=\lim_{n\to\infty}\frac{n+1}{n(n+2)}=0$$

so, this converges by the Ratio Test. In fact, you can (and should show):

$$s_n=\sum_{k=1}^{n}a_k=\frac{(n+1)!-1}{(n+1)!}\Rightarrow\sum_{k=1}^{\infty}\frac{n}{(n+1)!}=\lim_{n\to\infty}s_n=1$$

Problems

$$2+\frac{3}{2^3}+\frac{4}{3^3}+\frac{5}{4^3}+\cdots+\frac{n+1}{n^3}+\cdots$$

This converges by the Limit Comparison Test with the series

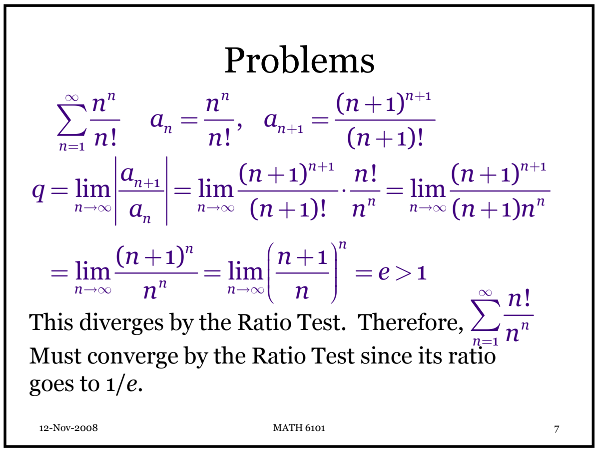
$$\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Problems

$$1+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n^n}$$

This converges by the Root Test.

$$\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\frac{1}{n}=0<1$$



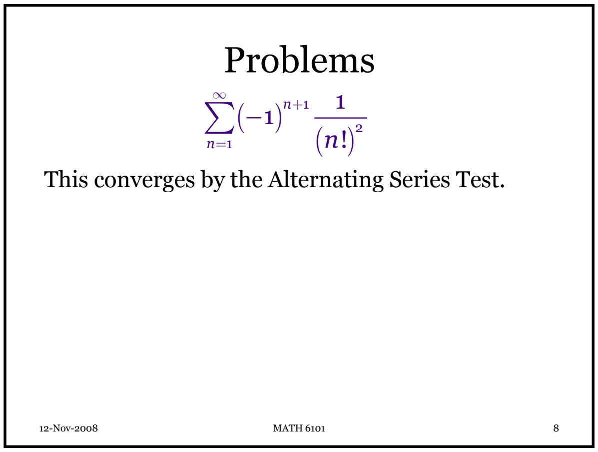
Problems

$$\sum_{n=1}^{\infty}\frac{n^n}{n!}\qquad a_n=\frac{n^n}{n!},\quad a_{n+1}=\frac{(n+1)^{n+1}}{(n+1)!}$$

$$q=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n}=\lim_{n\to\infty}\frac{(n+1)^{n+1}}{(n+1)n^n}$$

$$=\lim_{n\to\infty}\frac{(n+1)^n}{n^n}=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n=e>1$$

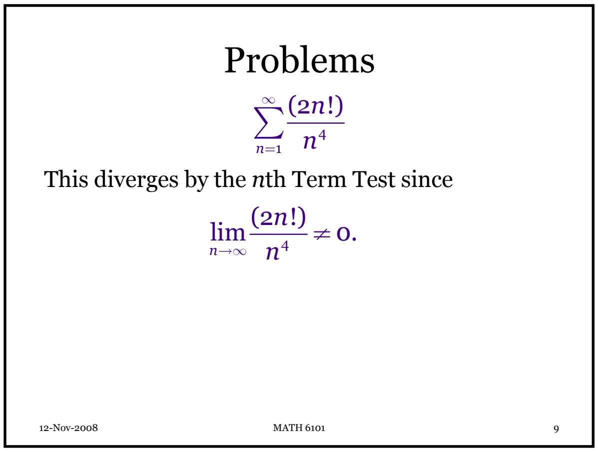
This diverges by the Ratio Test. Therefore, $\sum_{n=1}^{\infty}\frac{n!}{n^n}$ Must converge by the Ratio Test since its ratio goes to $1/e$.



Problems

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{(n!)^2}$$

This converges by the Alternating Series Test.



Problems

$$\sum_{n=1}^{\infty}\frac{(2n!)}{n^4}$$

This diverges by the *n*th Term Test since

$$\lim_{n\to\infty}\frac{(2n!)}{n^4}\neq 0.$$

Problems

$$\sum_{n=0}^{\infty}\left(\frac{n}{n+1}\right)^{n}$$

This diverges by the *n*th Term Test since

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n}=\frac{1}{e}\neq 0.$$

Problems

$$\sum_{n=1}^{\infty} nr^{n}, \quad |r|<1$$

$$q=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(n+1)r^{n+1}}{nr^{n}}\right|=|r|\lim_{n\to\infty}\frac{n+1}{n}$$

$$=|r|<1$$

This converges by the Ratio Test.

Problems

$$\text{Show} \quad \sum_{n=1}^{\infty} nr^{n}=\frac{r}{(1-r)^{2}}$$

$$\frac{1}{1-r}=\sum_{n=0}^{\infty} r^{n}=1+r+r^{2}+r^{3}+\cdots$$

$$\left(\frac{1}{1-r}\right)^{2}=\left(\sum_{n=0}^{\infty} r^{n}\right)^{2}=\left(1+r+r^{2}+r^{3}+\cdots\right)^{2}$$

$$\frac{1}{(1-r)^{2}}=1\cdot 1+\left(1\cdot r+r\cdot 1\right)+\left(1\cdot r^{2}+r\cdot r+r^{2}\cdot 1\right)+\cdots$$

$$=1+2r+3r^{2}+4r^{3}+\cdots$$

$$\frac{r}{(1-r)^{2}}=r+2r^{2}+3r^{3}+4r^{4}+\cdots$$

Problems

Find $\sum_{n=1}^{\infty}\frac{n}{2^n}$

$$\sum_{n=1}^{\infty}\frac{n}{2^n}=\sum_{n=1}^{\infty}n\left(\frac{1}{2}\right)^n=\frac{1/2}{\left(1-1/2\right)^2}=2$$

Prime Numbers

Statement:
There are an infinite number of prime numbers.

Prime Numbers

Theorem: $\sum 1/p$ diverges.

Probability

Proposition:

If two positive integers are chosen independently and randomly, then the probability that they are relatively prime is $6/\pi^2$.

Riemann Zeta Function

In 1859 Reimann defined a differentiable function of a complex variable ζ(*s*) by

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots + \frac{1}{n^s} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Riemann knew the value of this function at certain values of *s*.

ζ(0) does not exist. (Why?)

ζ(1) does not exist. (Why?)

$\zeta(2) = \pi^2/6$

$\zeta(4) = \pi^4/90$

Riemann Zeta Function

Question: Where is $\zeta(s) = 0$?

It can be shown that when it has been extended to all complex numbers, except $\mathrm{Re}(s) = 1$, then it is trivially seen to be zero at the negative even integers.

Riemann proved that $\zeta(s) = 0$ when *s* falls inside the infinite strip bounded by the lines $x = 0$ and $x = 1$.

Prime Number Theorem

Theorem:
For every real number x let π(x) be the number of prime numbers less than x and let

$$Li(x) = \int_2^x \frac{dt}{\ln(t)}$$

Then

$$\lim_{x\to\infty}\frac{\pi(x)}{Li(x)} = 1$$

This was proven by Hadamard and Poussin in 1896.

Riemann Zeta Hypothesis

Conjecture: (***Unproven***)
If s is a complex number so that $\zeta(s) = 0$ *then* $\mathrm{Re}(s) = 1/2$.

What we do know:

- The line x = ½ contains an infinite number of zeroes of $\zeta(s)$.
- The first 70,000,000 or so lie on that line.

Power Series

Definition:

If $\{a_n\}$ is a sequence, we define the series $\sum a_n x^n$ as a *power series* in *x*. For a given sequence a power series is a function $f(x)$ whose domain consists of those values of *x* for which the series converges.

Power series behavior is typical to that of the geometric series.

$\sum x^n$ converges for $|x|<1$, so the domain of this function: $f(x) = \sum x^n$ is the open interval (-1,1).

Convergence of Power Series

Proposition:

Suppose that the power series $\sum a_n x^n$ *converges for* $x = x'$ *and diverges for* $x = x''$, *then* $\sum a_n x^n$

1. *converges absolutely for each x satisfying* $|x| < |x'|$;
2. *diverges for each x satisfying* $|x| > |x''|$

Convergence of Power Series

Proposition:

Let $\sum a_n x^n$ *be a power series, then one of the following must hold.*

1. $\sum a_n x^n$ *converges absolutely for all x;*
2. $\sum a_n x^n$ *converges only at* $x = 0$;
3. *there is a number* $\rho > 0$ *so that* $\sum a_n x^n$ *converges absolutely for* $|x| < \rho$ *and diverges for* $|x| > \rho$.

Convergence of Power Series

ρ is the *radius of convergence* for the power series.

$S = [-\rho, \rho]$, $(-\rho, \rho]$, $[-\rho, \rho)$, or $(-\rho, \rho)$ and all can occur. This is called the *interval of convergence*. In Case 1, $\rho = \infty$ and $S = \mathbf{R}$; Case 2, $\rho = 0$ and $S = \{0\}$.

Ratio Test for Power Series

Proposition:

Let $\sum a_n x^n$ *be a power series with radius of convergence* ρ. *If* $a_n \neq 0$ *for all* n *and*

$$\{a_{n+1}/a_n\} \to q$$

then

1. $\rho = \infty$ *if* $q = 0$;
2. $\rho = 0$ *if* $q = \infty$;
3. $\rho = 1/|q|$ *otherwise.*

Proof

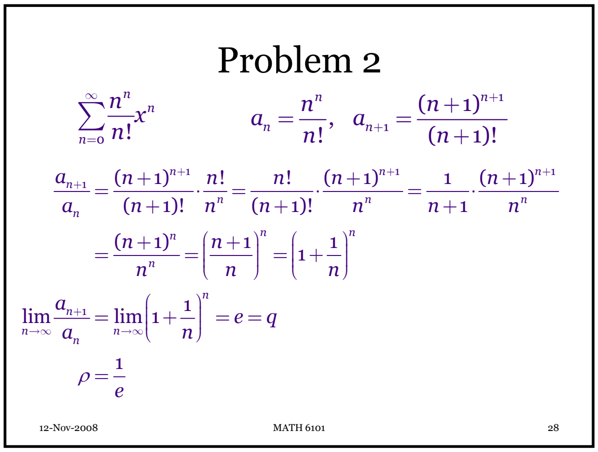
Problem 1

$$\sum_{n=1}^{\infty} \frac{x^n}{n(n+4)} \qquad a_n = \frac{1}{n(n+4)}, \quad a_{n+1} = \frac{1}{(n+1)(n+5)}$$

$$\frac{a_{n+1}}{a_n} = \frac{1}{(n+1)(n+5)} \cdot \frac{n(n+4)}{1} = \frac{n(n+4)}{(n+1)(n+5)}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n(n+4)}{(n+1)(n+5)} = 1 = q$$

$$\rho = \frac{1}{q} = 1$$



Problem 2

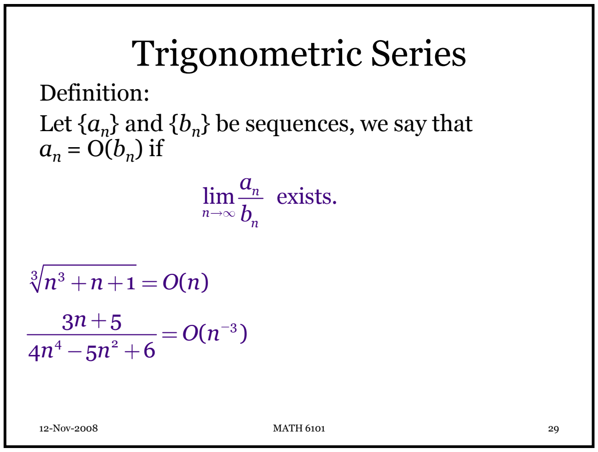
$$\sum_{n=0}^{\infty}\frac{n^n}{n!}x^n \qquad a_n=\frac{n^n}{n!},\quad a_{n+1}=\frac{(n+1)^{n+1}}{(n+1)!}$$

$$\frac{a_{n+1}}{a_n}=\frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n}=\frac{n!}{(n+1)!}\cdot\frac{(n+1)^{n+1}}{n^n}=\frac{1}{n+1}\cdot\frac{(n+1)^{n+1}}{n^n}$$

$$=\frac{(n+1)^n}{n^n}=\left(\frac{n+1}{n}\right)^n=\left(1+\frac{1}{n}\right)^n$$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e=q$$

$$\rho=\frac{1}{e}$$



Trigonometric Series

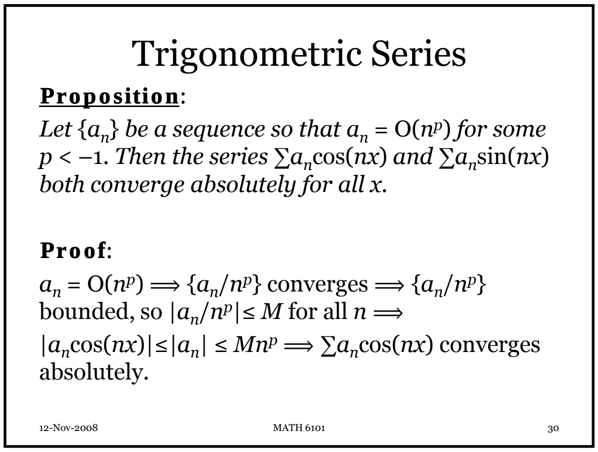
Definition:

Let $\{a_n\}$ and $\{b_n\}$ be sequences, we say that $a_n = \mathrm{O}(b_n)$ if

$$\lim_{n\to\infty}\frac{a_n}{b_n}\ \text{ exists.}$$

$$\sqrt[3]{n^3+n+1}=O(n)$$

$$\frac{3n+5}{4n^4-5n^2+6}=O(n^{-3})$$



Trigonometric Series

Proposition:

Let $\{a_n\}$ *be a sequence so that* $a_n = \mathrm{O}(n^p)$ *for some* $p < -1$*. Then the series* $\sum a_n\cos(nx)$ *and* $\sum a_n\sin(nx)$ *both converge absolutely for all* x*.*

Proof:

$a_n = \mathrm{O}(n^p) \Longrightarrow \{a_n/n^p\}$ converges $\Longrightarrow \{a_n/n^p\}$ bounded, so $|a_n/n^p| \le M$ for all $n \Longrightarrow$ $|a_n\cos(nx)| \le |a_n| \le Mn^p \Longrightarrow \sum a_n\cos(nx)$ converges absolutely.