



MATH 6101
Fall 2008

Continuity

Continuity

Euler (1748) defined continuous, discontinuous and mixed functions.

Continuous = expressible by a single analytic expression.

Mixed = expressible in two or more analytic expressions.

Discontinuous = defined by different expression at different places

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Continuity

Fourier (1807) - *On the Propagation of Heat in Solid Bodies*


Committee of Lagrange, Laplace, Monge and Lacroix was set up to report on the work.

Memoir highly regarded but caused controversy.

I. First objection (Lagrange and Laplace, 1808): Fourier's expansions of functions as trigonometrical series

II. Second objection (Biot): Fourier's derivation of equations of transfer of heat.

Théorie analytique de la chaleur (1822)



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Continuity

Bolzano (1817) *Rein analytischer Beweis*
(Pure Analytical Proof)

Attempt to free calculus from concept of
infinitesimal

Bolzano achieved what he set out to achieve
His ideas only became well known after his
death – almost 100 years.

Bolzano purged the concepts of limit,
convergence, and derivative of geometrical
components and replace them by purely
arithmetical concepts

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Continuity

Bolzano (1817) *Rein analytischer Beweis*
(Pure Analytical Proof)

Defined a function f to be **continuous** on
an interval if for any value of x in this
interval the difference $f(x+\Delta x) - f(x)$
becomes and remains less than any given
quantity for Δx sufficiently small, whether
positive or negative.

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Continuity

Cauchy (1821) – *Cours d'analyse*

Let f be a function that maps a set of real
numbers to another set of real numbers, and
suppose c is an element of the domain of f .
The function f is said to be **continuous** at
the point c if the following holds: For any
number $\varepsilon > 0$, however small, there exists
some number $\delta > 0$ such that for all x in the
domain with $c - \delta < x < c + \delta$, the value of f
(x) satisfies

$$f(c) - \varepsilon < f(x) < f(c) + \varepsilon$$

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Continuity

Cauchy (1821) – *Cours d'analyse*

Pointed out Euler's definition of continuity was imprecise

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases} = \sqrt{x^2} = \frac{2}{\pi} \int_0^{\infty} \frac{x^2}{t^2 + x^2} dt$$

The first is discontinuous by Euler, but last two are clearly analytic expressions.

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Continuity

Heine – 1860's

A real function f is **continuous** if for any sequence $\{x_n\}$ such that $\lim_{n \rightarrow \infty} x_n = L$ it holds that $\lim_{n \rightarrow \infty} f(x_n) = f(L)$.

(Assume $\{x_n\}$ & L are in domain of f .)

A function is continuous if and only if it preserves limits. (Cauchy's and Heine's definitions of continuity are equivalent on the reals.)

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Limits

Let $f: D \rightarrow \mathbf{R}$ be a function. Let $a \in D$.

Definition 1: $\lim_{x \rightarrow a} f(x) = L$ provided

(1) There is a sequence $\{x_n\} \subset D - \{a\}$ such that $\lim_{n \rightarrow \infty} x_n = a$, and

(2) for every sequence $\{x_n\} \subset D - \{a\}$ such that $\lim_{n \rightarrow \infty} x_n = a$, $\lim_{n \rightarrow \infty} f(x_n) = L$.

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Simple Proposition

Proposition:

(1) $\lim_{x \rightarrow a} r = r$; (2) $\lim_{x \rightarrow a} x = a$; (3) $\lim_{x \rightarrow a} |x| = |a|$.

Big Theorem:

Suppose $f, g: D \rightarrow \mathbf{R}$ so that $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$, then

(1) $\lim_{x \rightarrow a} (f(x) + g(x)) = L + M$

(2) $\lim_{x \rightarrow a} f(x)g(x) = LM$

(3) $\lim_{x \rightarrow a} f(x) - g(x) = L - M$

(4) $\lim_{x \rightarrow a} f(x)/g(x) = L/M$ if $M \neq 0$.

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Problems

$$\lim_{x \rightarrow 5} \frac{x-5}{x^2-25}$$

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Problems

$$\lim_{x \rightarrow 4} \frac{4}{x-4}$$

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Problems

$$\lim_{x \rightarrow 0} \frac{x}{|x|}$$

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The Dirichlet Function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

$\lim_{x \rightarrow a} f(x)$ does not exist for all $a \in \mathbb{R}$

Let $g(x) = x \cdot f(x)$

Claim: $\lim_{x \rightarrow 0} g(x) = 0$

but $\lim_{x \rightarrow a} g(x)$ does not exist for all $a \neq 0$.

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Continuity

Definition:

The function $f: D \rightarrow \mathbf{R}$ is **continuous** at $a \in D$ if .

$$\lim_{x \rightarrow a} f(x) = f(a)$$

This means:

For a function to be continuous at a point a :

1. $f(a)$ exists,
2. $\lim_{x \rightarrow a} f(x)$ exists, and
3. $\lim_{x \rightarrow a} f(x) = f(a)$.

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Simple Proposition & Big Theorem

Proposition: Redux

The constant function $f(x) = r$, the identity function $f(x) = x$ and the absolute value function $f(x) = |x|$ are all continuous for all real numbers.

Big Theorem:

Suppose $f, g: D \rightarrow \mathbf{R}$ are continuous at $x = a$. Then

- (1) $f(x) + g(x)$ is continuous at $x = a$.
- (2) $f(x)g(x)$ is continuous at $x = a$.
- (3) $f(x) - g(x)$ is continuous at $x = a$.
- (4) $f(x)/g(x)$ is continuous at $x = a$ if $g(a) \neq 0$.

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Continuity and Composition

Theorem:

Suppose $f: D \rightarrow \mathbf{R}$ and $g: E \rightarrow \mathbf{R}$ are functions such that the composition $f \circ g$ is defined in E . If g continuous at $x = a \in E$ and f continuous at $g(a)$, then $f \circ g$ is continuous at $x = a$.

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Continuity and Trig

Theorem:

1. $|\sin \phi| \leq |\phi|$ for all ϕ
2. $\lim_{\phi \rightarrow 0} \frac{\sin \phi}{\phi} = 1$

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Continuity and Trig

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Continuity and Trig

Theorem:
The functions $\sin(x)$ and $\cos(x)$ are continuous.

$$\sin \alpha - \sin \beta = 2 \sin \frac{\alpha - \beta}{2} \cos \frac{\alpha + \beta}{2}$$

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Questions

Let $\{f_n\}$ be a sequence of continuous functions. By induction we know that

$$\sum_{k=1}^n f_k(x)$$

is continuous. Is it true that the following sum is continuous?

$$\sum_{n=1}^{\infty} f_n(x)$$

Is it true that if $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ exists, then it will be continuous?

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Continuity Pathologies

Let

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Let $g(x) = x \cdot f(x)$

Then g is continuous at $x = 0$ and discontinuous at every other real number.

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Continuity II

Determine where the following function is continuous.

$$f(x) = x + \lceil x^2 \rceil - \lfloor x \rfloor$$

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More Pathologies

1. A function that is continuous at only one point ($x=0$)

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ -x & \text{if } x \text{ is irrational} \end{cases}$$

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More Pathologies

3. A function with a derivative defined for all x , but whose derivative is discontinuous

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x=0 \end{cases}$$

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More Pathologies

4. A function continuous at all irrationals and discontinuous at all rationals.

$f(x)=1/q$ if $x = p/q$ is rational and in lowest terms, otherwise $f(x)=0$.

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More Pathologies

5. The function g is non-zero, infinitely differentiable, and any derivative of g at $x = 0$ equals zero.

$$g(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x=0 \end{cases}$$

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More Pathologies

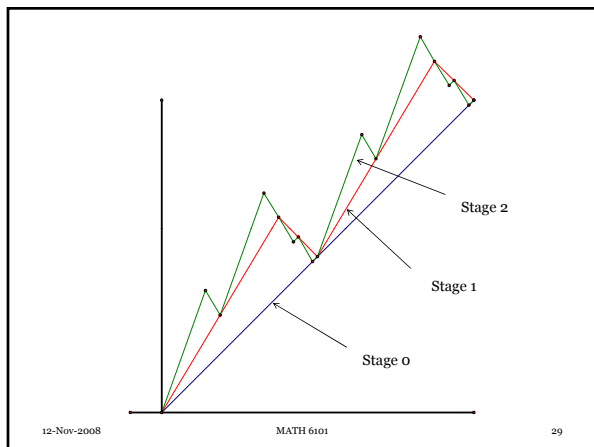
7. A function that is everywhere continuous but nowhere differentiable (Bolzano)
- a) The first graph is the line from (0, 0) to (0, 1).
 - b) Suppose that (a,A) and (b,B) are the endpoints of a segment in some iteration. In the next iteration the segment is replaced by a polygonal line joining the following points:
 (a,A) , $(a+3/8(b-a), A+5/8(B-A))$,
 $(1/2(a+b), 1/2(A+B))$, $(a+7/8(b-a), A+9/8(B-A))$,
 (b,B)

<http://demonstrations.wolfram.com/BolzosFunction/>

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More Pathologies

7. A function that is everywhere continuous but nowhere differentiable (Weierstrass, 1872)



$$g(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n |\sin(4^n x)|$$

$$h(x) = \sum_{n=0}^{\infty} (1/2)^n \sin(2^n x)$$

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Properties of Continuous Functions

Intermediate Value Theorem (Bolzano, 1817)

Let $f:[a,b] \rightarrow \mathbf{R}$ be a continuous function and let y^* be a real number so that either $f(a) < y^* < f(b)$ or $f(b) < y^* < f(a)$. Then there is x^* , $a < x^* < b$, so that $f(x^*) = y^*$.

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Properties of Continuous Functions

Corollary: For every positive integer n and positive real number r there is a real number x so that $x^n = r$.

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Properties of Continuous Functions

The Maximum Principle (Weierstrauss):
Let $f:[a,b] \rightarrow \mathbf{R}$ be a continuous function. Then f assumes both a maximum and a minimum on $[a,b]$.

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