MATH 6101
Fall 2008

Continuity

Continuity

Euler (1748) defined continuous, discontinuous and mixed functions.

Continuous = expressible by a single analytic expression.

Mixed = expressible in two or more analytic expressions.

Discontinuous = defined by different expression at different places

Continuity

Fourier (1807) - *On the Propagation of Heat in Solid Bodies*

Committee of Lagrange, Laplace, Monge and Lacroix was set up to report on the work.

Memoir highly regarded but caused controversy.

I. First objection (Lagrange and Laplace, 1808): Fourier's expansions of functions as trigonometrical series

II. Second objection (Biot): Fourier's derivation of equations of transfer of heat.

Théorie analytique de la chaleur (1822)

Continuity

- Bolzano (1817) *Rein analytischer Beweis* (Pure Analytical Proof)
- Attempt to free calculus from concept of infinitesimal
- Bolzano achieved what he set out to achieve
- His ideas only became well known after his death – almost 100 years.
- Bolzano purged the concepts of limit, convergence, and derivative of geometrical components and replace them by purely arithmetical concepts

Continuity

Bolzano (1817) *Rein analytischer Beweis* (Pure Analytical Proof)

Defined a function f to be ***continuous*** on an interval if for any value of x in this interval the difference $f(x+\Delta x) - f(x)$ becomes and remains less than any given quantity for Δx sufficiently small, whether positive or negative.

Continuity

Cauchy (1821) – *Cours d'analyse*

Let f be a function that maps a set of real numbers to another set of real numbers, and suppose c is an element of the domain of f. The function f is said to be ***continuous*** at the point c if the following holds: For any number $\varepsilon > 0$, however small, there exists some number $\delta > 0$ such that for all x in the domain with $c - \delta < x < c + \delta$, the value of $f(x)$ satisfies

$$f(c) - \varepsilon < f(x) < f(c) + \varepsilon$$

Continuity

Cauchy (1821) – *Cours d'analyse*

Pointed out Euler's definition of continuity was imprecise

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases} = \sqrt{x^2} = \frac{2}{\pi}\int_0^\infty \frac{x^2}{t^2 - x^2}\,dt$$

The first is discontinuous by Euler, but last two are clearly analytic expressions.

Continuity

Heine – 1860's

A real function f is ***continuous*** if for any sequence $\{x_n\}$ such that $\lim_{n\to\infty} x_n = L$ it holds that $\lim_{n\to\infty} f(x_n) = f(L)$.

(Assume $\{x_n\}$ & L are in domain of f.)

A function is continuous if and only if it preserves limits. (Cauchy's and Heine's definitions of continuity are equivalent on the reals.)

Limits

Let $f : D \to \boldsymbol{R}$ be a function. Let $a \in D$.

Definition 1: $\lim_{x \to a} f(x) = L$ provided

(1) There is a sequence $\{x_n\} \subset D - \{a\}$ such that $\lim_{n \to \infty} x_n = a$, and

(2) for every sequence $\{x_n\} \subset D - \{a\}$ such that $\lim_{n \to \infty} x_n = a$, $\lim_{n \to \infty} f(x_n) = L$.

Simple Proposition

Proposition:

(1) $\lim_{x\to a} r = r$; (2) $\lim_{x\to a} x = a$; (3) $\lim_{x\to a}|x| = |a|$.

Big Theorem:

Suppose $f,g: D \to \boldsymbol{R}$ so that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then

(1) $\lim_{x\to a} (f(x) + g(x)) = L + M$

(2) $\lim_{x\to a} f(x)g(x) = LM$

(3) $\lim_{x\to a} f(x) - g(x) = L - M$

(4) $\lim_{x\to a} f(x)/g(x) = L/M$ if $M \neq 0$.

Problems

$$\lim_{x \to 5} \frac{x-5}{x^2-25}$$

Problems

$$\lim_{x \to 4} \frac{4}{x-4}$$

Problems

$$\lim_{x \to 0} \frac{x}{|x|}$$

The Dirichlet Function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

$\lim_{x \to a} f(x)$ does not exist for all $a \in R$

Let $g(x) = x \cdot f(x)$

Claim: $\lim_{x \to 0} g(x) = 0$

but $\lim_{x \to a} g(x)$ does not exist for all $a \neq 0$.

Continuity

Definition:

The function $f : D \to \boldsymbol{R}$ is ***continuous*** at $a \in$ D if .

$$\lim_{x \to a} f(x) = f(a)$$

This means:

For a function to be continuous at a point a:

1. $f(a)$ exists,
2. $\lim_{x \to a} f(x)$ exists, and
3. $\lim_{x \to a} f(x) = f(a)$.

Simple Proposition & Big Theorem Redux

Proposition:

The constant function $f(x) = r$, the identity function $f(x) = x$ and the absolute value function $f(x) = |x|$ are all continuous for all real numbers.

Big Theorem:

Suppose $f,g: D \to \boldsymbol{R}$ are continuous at $x = a$. Then

(1) $f(x) + g(x)$ is continuous at $x = a$.

(2) $f(x)g(x)$ is continuous at $x = a$.

(3) $f(x) - g(x)$ is continuous at $x = a$.

(4) $f(x)/g(x)$ is continuous at $x = a$ if $g(a) \neq 0$.

Continuity and Composition

Theorem:

Suppose f: $D \to \boldsymbol{R}$ and g: $E \to \boldsymbol{R}$ are functions such that the composition $f \circ g$ is defined in E. If g continuous at $x = a \in E$ and f continuous at $g(a)$, then $f \circ g$ is continuous at $x = a$.

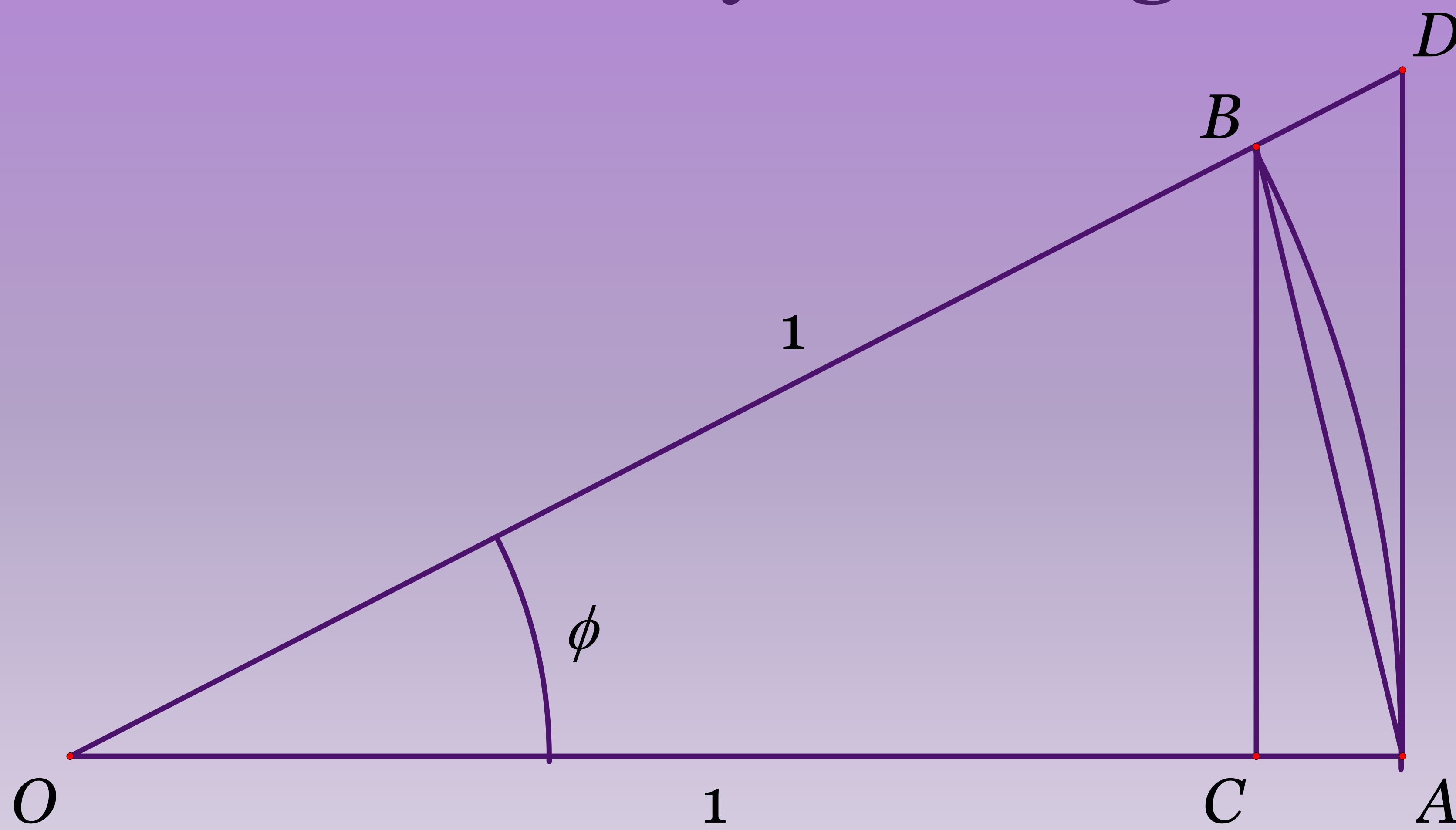
Continuity and Trig

***Theorem*:**

1. $\left|\sin\phi\right| \leq \left|\phi\right|$ for all ϕ

2. $\lim\limits_{\phi\to 0}\dfrac{\sin\phi}{\phi} = 1$

Continuity and Trig



Continuity and Trig

Theorem:

The functions sin(x) and cos(x) are continuous.

$$\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$$

$$\left|\sin x_n - \sin a\right| \le 2\left|\sin\frac{x_n - a}{2}\right| \le 2\frac{\left|x_n - a\right|}{2} = \left|x_n - a\right|$$

Questions

Let $\{f_n\}$ be a sequence of continuous functions. By induction we know that

$$\sum_{k=1}^{n} f_k(x)$$

is continuous. Is it true that the following sum is continuous?

$$\sum_{n=1}^{\infty} f_n(x)$$

Is it true that if $f(x) = \lim_{n\to\infty} f_n(x)$ exists, then it will be continuous?

Continuity Pathologies

Let

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

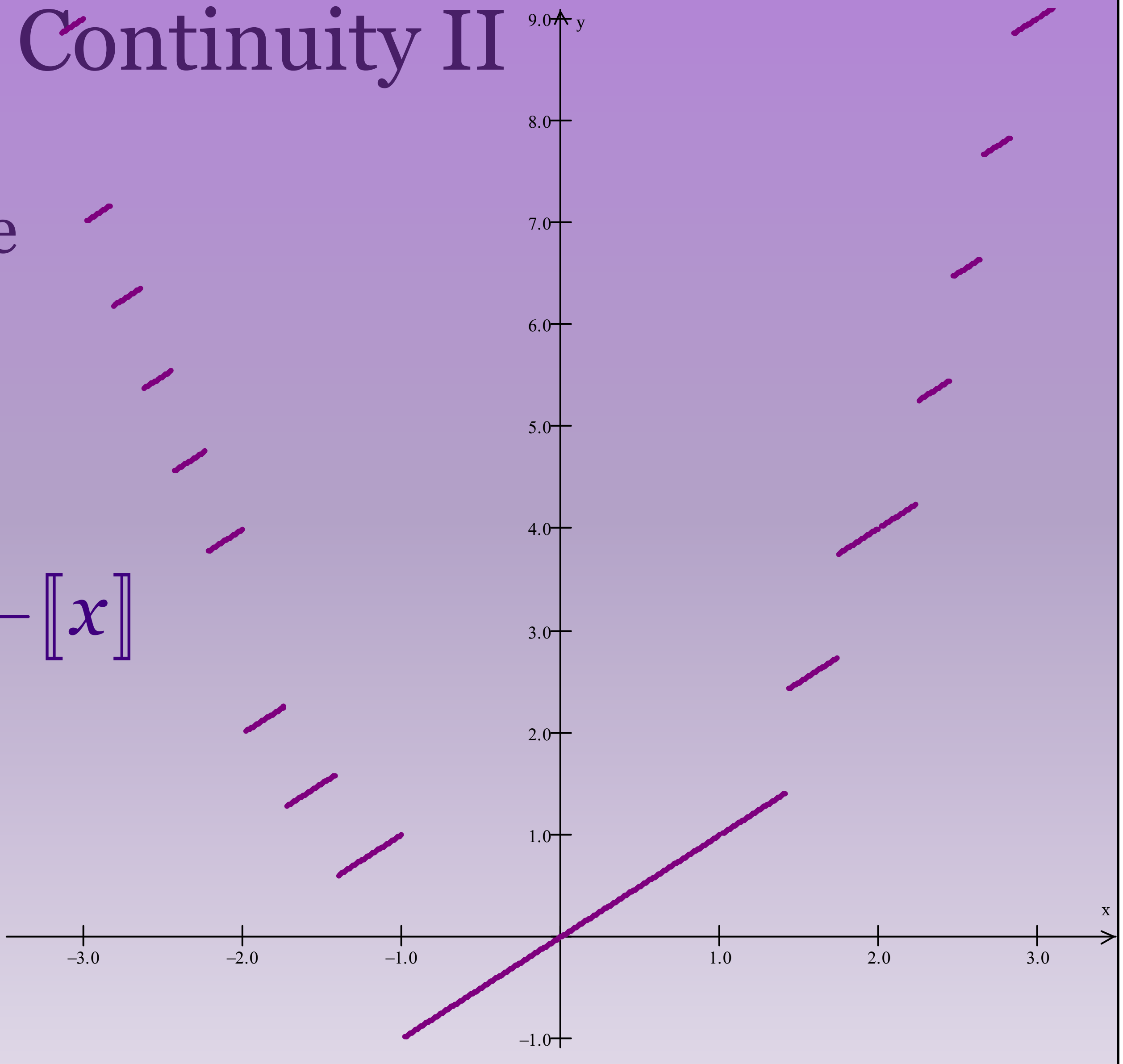
Let $g(x) = x \cdot f(x)$

Then g is continuous at $x = 0$ and discontinuous at every other real number.

Continuity II

Determine where the following function is continuous.

$$f(x) = x + [\![x^2]\!] - [\![x]\!]$$



Continuity II

The function, $f(x)$, is a combination of three simpler functions.

$f_1(x) = x$ is continuous at each point;

$f_2(x) = [\![x]\!]$ is continuous $\Leftrightarrow$ x is not an integer;

$f_3(x) = [\![x^2]\!]$ is continuous $\Leftrightarrow$ x^2 is not an integer;

More Pathologies

1. A function that is continuous at only one point (x=0)

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ -x & \text{if } x \text{ is irrational} \end{cases}$$

More Pathologies

2. A function with a derivative defined for all x, but whose derivative is discontinuous

$$f(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

More Pathologies

3. A function continuous at all irrationals and discontinuous at all rationals.

   $f(x)=1/q$ if $x = p/q$ is rational and in lowest terms, otherwise $f(x)=0$.

More Pathologies

4. The function g is non-zero, infinitely differentiable, and any derivative of g at x = 0 equals zero.

$$g(x)=\begin{cases} e^{-1/x^2} & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$$

More Pathologies

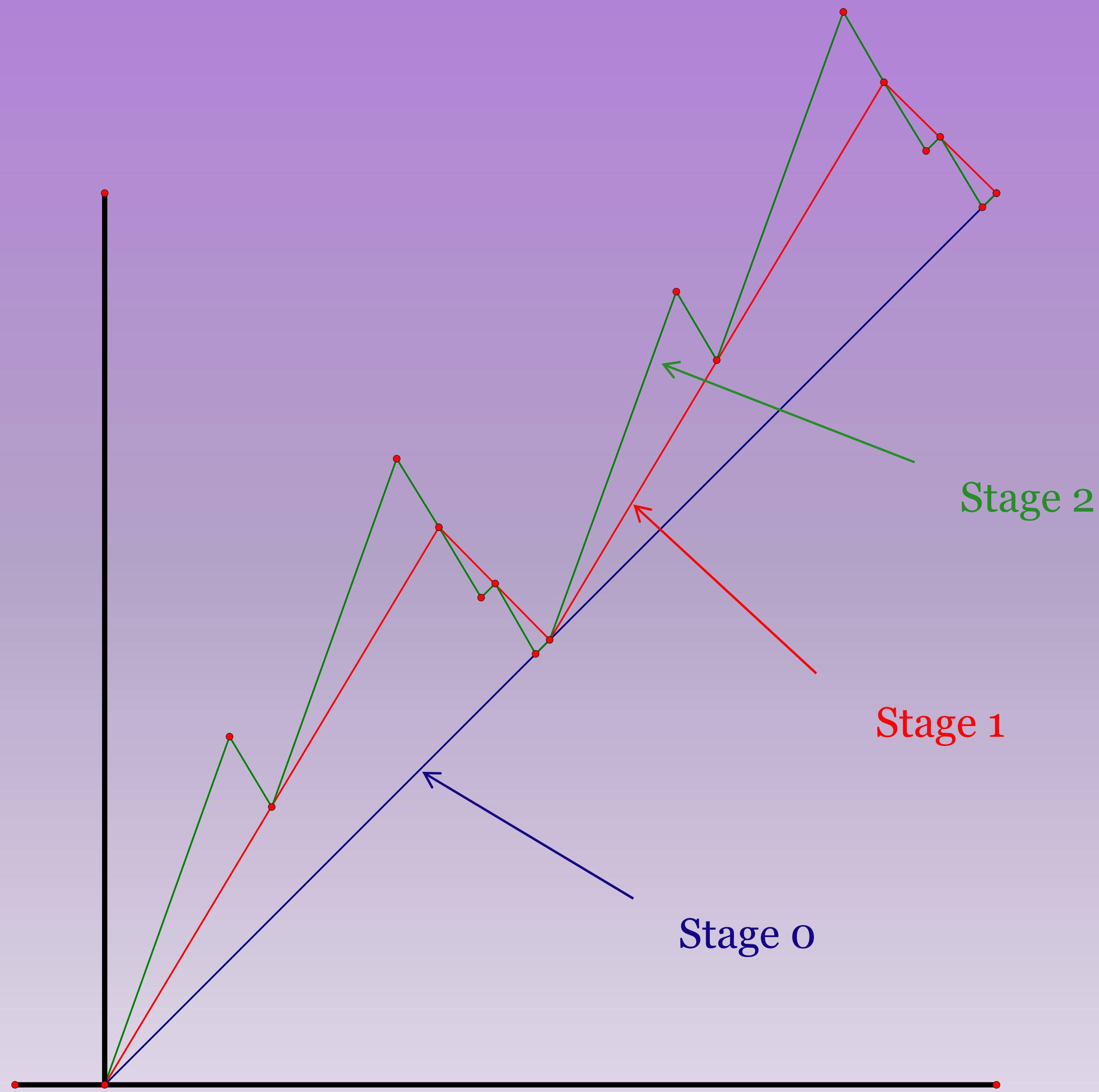
5. A function that is everywhere continuous but nowhere differentiable (Bolzano)

a) The first graph is the line from (0, 0) to (0, 1).

b) Suppose that (a,A) and (b,B) are the endpoints of a segment in some iteration. In the next iteration the segment is replaced by a polygonal line joining the following points:

(a,A), $(a+3/8(b-a),A+5/8(B-A))$, $(1/2(a+b),1/2(A+B))$, $(a+7/8(b-a),A+9/8(B-A))$, (b,B)

http://demonstrations.wolfram.com/BolzanosFunction/


Stage 2
Stage 1
Stage 0

More Pathologies

7. A function that is everywhere continuous but nowhere differentiable (Weierstrass, 1872)

$$g(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \left|\sin(4^n x)\right|$$

$$h(x) = \sum_{n=0}^{\infty} (1/2)^n \sin(2^n x)$$

Properties of Continuous Functions

Intermediate Value Theorem (Bolzano, 1817)

Let $f:[a,b] \to \boldsymbol{R}$ *be a continuous function and let* y^* *be a real number so that either* $f(a) < y^* < f(b)$ *or* $f(b) < y^* < f(a)$. *Then there is* x^*, $a < x^* < b$, *so that* $f(x^*) = y^*$.

Properties of Continuous Functions

Proof:

Assume $f(a) < y^* < f(b)$.

We are going to set up a binary search for x^*.

Let $a_0 = a$ and $b_0 = b$. Let $c_1 = (a_0 + b_0)/2$

Then $f(c_1) = y^*$, $f(c_1) < y^*$, or $f(c_1) > y^*$.

Properties of Continuous Functions

If $f(c_1) > y^*$ set $a_1 = a_0$ and $b_1 = c_1$.

If $f(c_1) < y^*$ set $a_1 = c_1$ and $b_1 = b_0$.

If $f(c_1) = y^*$ set $x^* = c_1$.

This sets up a recursive algorithm.

If there is an n so that $f(c_n) = y^*$, we are done.

Properties of Continuous Functions

Assume that this never happens.

Note that

$a_0 \le a_1 \le a_2 \le \ldots$ and $b_0 \ge b_1 \ge b_2 \ge \ldots$ and

$$b_n - a_n = \frac{b_{n-1} - a_{n-1}}{2} = \frac{b_{n-2} - a_{n-2}}{2^2} = \cdots = \frac{b_0 - a_0}{2^n}$$

Therefore $\{a_n\}$ and $\{b_n\}$ converge to the same limit, call it x^*.

f continuous $\Longrightarrow \{f(a_n)\}$ and $\{f(b_n)\}$ converge to $f(x^*)$. By the Squeeze Theorem

Properties of Continuous Functions

$$f(x^*) = \lim_{n\to\infty} f(a_n) \le y^* \le \lim_{n\to\infty} f(b_n) = f(x^*)$$

Therefore $f(x^*) = y^*$.

Properties of Continuous Functions

<u>Corollary:</u> *For every positive integer n and positive real number r there is a real number x so that* $x^n = r$.

Proof:

Let $f(x) = x^n$, $y^* = r$, $a = 0$, and $b = 1+r$.

$$f(a) = 0 < y^* = r < 1 + r < (1+r)^n = f(b)$$

Properties of Continuous Functions

The Maximum Principle (Weierstrauss):
Let f:[a,b] → $\boldsymbol{R}$ *be a continuous function. Then f assumes both a maximum and a minimum on* [a,b].