5.2 Geodesic Equations and Clairaut Relation

Now that we know what geodesics are, how can we do the calculations? Let's consider an orthogonal patch $\mathbf{x}(u,v)$, (i.e., $F = \mathbf{x}_u \cdot \mathbf{x}_v = 0$). Let σ be a geodesic in the patch $\mathbf{x}$. Then, we know that we can write $\sigma(t)$ as $\sigma(t) = \mathbf{x}(u(t), v(t))$ and it follows that $\sigma' = \mathbf{x}_u u' + \mathbf{x}_v v'$ and

$$\sigma'' = \mathbf{x}_{uu} {u'}^2 + \mathbf{x}_{uv} v' u' + \mathbf{x}_u u'' + \mathbf{x}_{vu} u' v' + \mathbf{x}_{vv} {v'}^2 + \mathbf{x}_v v''$$

Now, from our previous work, we can replace $\mathbf{x}_{uu}$, $\mathbf{x}_{uv}$, and $\mathbf{x}_{vv}$ by their representations in $\{\mathbf{x}_u, \mathbf{x}_v, U\}$. We find

$$\begin{aligned}\sigma'' = \mathbf{x}_u \left[u'' + \frac{E_u}{2E}{u'}^2 + \frac{E_v}{E}u'v' - \frac{G_u}{2E}{v'}^2\right] \\ + \mathbf{x}_v \left[v'' - \frac{E_v}{2E}{u'}^2 + \frac{G_u}{G}u'v' - \frac{G_v}{2G}{v'}^2\right] \\ + U\left[l{u'}^2 + 2mu'v' + n{v'}^2\right]\end{aligned}$$

So the first two terms give us the tangential part of σ''. For σ to be a geodesic, these terms must be zero, so it is necessary and sufficient that the following *geodesic equations* are satisfied.

$$\begin{aligned} u'' + \frac{E_u}{2E}{u'}^2 + \frac{E_v}{E}u'v' - \frac{G_u}{2E}{v'}^2 &= 0 \\ v'' - \frac{E_v}{2E}{u'}^2 + \frac{G_u}{G}u'v' - \frac{G_v}{2G}{v'}^2 &= 0 \end{aligned} \qquad (\textbf{Geodesic Equations})$$

The geodesic equations are a system of 2nd order differential equations.

Theorem 5.1 *Let $p = \mathbf{x}(u_0, v_0)$ be a point on a surface M with patch $\mathbf{x}(u,v)$ and let $\mathbf{v} \in T_pM$. Then there is a unique geodesic $\sigma\colon (-r, r) \to M$ with $\sigma(0) = p$ and $\sigma'(0) = \mathbf{v}$.*

The proof of this is based on the theory of differential equations, and the existence of a solution to a system with certain intial values. Our geodesic equations satisfy the hypothesis of the theorem, so such a solution exists.

Let's look at an example.

Example 5.3 The Unit Sphere, $\mathbf{S}^2$. Take the standard patch for the unit sphere.

$$\mathbf{x}(u,v) = (\cos u \cos v, \sin u \cos v, \sin v)$$

We have calculated previously that $E = \cos^2 v$, $F = 0$, and $G = 1$. The geodesic equations for the unit sphere become

$$u'' - 2\tan v \; v' \; u' = 0 \qquad\qquad v'' + \sin v \cos v \; {u'}^2 = 0.$$

How in the world will we ever be able to solve such a system of 2nd order non-linear differential equations?!? In fact, it is the case that we usually CANNOT solve the differential equations directly. In the case of the sphere, though, we have a few tricks up our collective sleeves to bring to bear on the problem.

1. Assume that $\sigma(t) = \mathbf{x}(u(t), v(t))$ is a unit speed geodesic. Then since $\sigma' = u'\mathbf{x}_u + v'\mathbf{x}_v$ we have the unit speed relation

$$Eu'^2 + Gv'^2 = 1$$

   which on the sphere is the condition

$$\cos^2 v \; u'^2 + v'^2 = 1.$$

2. Solve the first geodesic equation as follows:

$$\begin{aligned} \int \frac{u''}{u'} &= \int 2\tan v \; v' \\ \ln(u') &= -2\ln(\cos v) + C \\ u' &= \frac{c}{\cos^2 v} \end{aligned}$$

3. Now replace u' in the unit speed equation by $\dfrac{c}{\cos^2 v}$

$$\begin{aligned} 1 &= \frac{c^2}{\cos^4 v}\cos^2 v + v'^2 \\ v'^2 &= 1 - \frac{c^2}{\cos^2 v} \\ v' &= \sqrt{\frac{\cos^2 v - c^2}{\cos^2 v}} \end{aligned}$$

4. Now, divide u' by v' and we get a separable differential equation

$$\frac{du}{dv} = \frac{c}{\cos v\sqrt{\cos^2 v - c^2}}.$$

5. Integrate this to find

$$\begin{aligned} u &= \int \frac{c}{\cos v\sqrt{\cos^2 v - c^2}} dv \\ &= \int \frac{c\sec^2 v}{\sqrt{1 - c^2\sec^2 v}} dv \\ &= \int \frac{c\sec^2 v}{\sqrt{1 - c^2 - c^2\tan^2 v}} dv \\ &= \int \frac{dw}{\sqrt{1 - w^2}} \qquad \text{substitute } w = \frac{c\tan v}{\sqrt{1-c^2}} \\ &= \arcsin(w) + d = \arcsin\left(\frac{c\tan v}{\sqrt{1-c^2}}\right) + d \end{aligned}$$

   Thus,

$$\sin(u - d) = \lambda\tan v$$

   where $\lambda = \frac{c}{\sqrt{1-c^2}}$. Now, use the Angle Sum formula for Sines to get $\sin(u - d) = \sin u\cos d - \sin d\cos u$.

6. This gives us

$$\sin u \cos d - \sin d \cos u - \lambda \tan v = 0$$
$$\frac{\sin u \cos v}{\cos v} \cos d - \sin d \frac{\cos u \cos v}{\cos v} - \frac{\lambda \sin v}{\cos v} = 0$$

Going back to our spherical coordinates $x = \cos u \cos v$, $y = \sin u \cos v$ and $z = \sin v$ and considering only the numerator gives

$$(-\sin d)x + (\cos d)y - \lambda z = 0$$

Thus, the geodesic equations imply that σ lies on a plane through the origin — just what we expected!

Let us look at yet another piece of information afforded to us by the geodesic equations. Let ϕ denote the smaller angle between the vectors σ' and $\mathbf{x}_v$ at any point along the the unit speed curve σ. The makes the angle between σ' and $\mathbf{x}_u$ $\frac{\pi}{2} - \phi$. Thus,

$$\sin(\phi) = \cos\left(\frac{\pi}{2} - \phi\right) = \frac{\sigma' \cdot \mathbf{x}_u}{\|\sigma'\|\|\mathbf{x}_u\|} = \frac{1}{\sqrt{E}}[(u'\mathbf{x}_u + v'\mathbf{x}_v) \cdot \mathbf{x}_u] = u'\sqrt{E} = u' \cos v$$

but $u' = \frac{c}{\cos^2 v}$ from the geodesic equations, so

$$\sin \phi = \frac{c}{\cos v}.$$

This is a special case of a result known as *Clairaut's relation*.

Example 5.4 The Torus. Take the torus with the parametrization

$$\mathbf{x}(u, v) = ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u).$$

We have previously computed $E = r^2$, $F = 0$, and $G = (R + r\cos u)^2$, so the geodesic equations are

$$u'' + \frac{(R + r\cos u)}{r} \sin u \, v'^2 = 0$$
$$v'' - 2\frac{r \sin u}{(R + r\cos u)} u'v' = 0$$

Then second equation is separable giving

$$v' = \frac{c}{(R + r\cos u)^2}.$$

Again, assume that σ has unit speed, replace v' in the unit speed relation by the above to get

$$1 = r^2 u'^2 + \frac{c^2}{(R + r\cos u)^2}$$
$$u' = \frac{1}{r}\sqrt{1 - \frac{c^2}{(R + r\cos u)^2}}$$

Dividing v' by u' we get

$$\begin{aligned} \frac{dv}{du} &= \frac{cr\sqrt{R+r\cos u}}{\sqrt{(R+r\cos u)^2-c^2}} \\ v &= \int \frac{cr\sqrt{R+r\cos u}}{\sqrt{(R+r\cos u)^2-c^2}}\,du \end{aligned}$$

We cannot integrate the right-hand side of this equation using elementary functions. This means that our Clairaut relation will have added importance.

We say that an orthogonal patch $\mathbf{x}(u,v)$ is a *Clairaut parametrization in* u if $E_v = 0$ and $G_v = 0$. The patch is *Clairaut in* v if $E_u = 0$ and $G_u = 0$. We have shown that the sphere is Clairaut in v and the torus is Clairaut in u. In these cases the geodesic equations simplify to:

$$\begin{aligned} u'' + \frac{E_u}{2E}u'^2 - \frac{G_u}{2E}v'^2 &= 0 \\ v'' + \frac{G_u}{G}u'v' &= 0 \end{aligned} \qquad (\textbf{Clairaut in } u)$$

$$\begin{aligned} u'' + \frac{E_v}{E}u'v' &= 0 \\ v'' - \frac{E_v}{2G}u'^2 + \frac{G_v}{2G}v'^2 &= 0 \end{aligned} \qquad (\textbf{Clairaut in } v)$$

The following theorem is stated for the u-Clairaut parametrizations, but everything also applies to the v-Clairaut case as well.

Theorem 5.2 *Let M be a surface with a u-Clairaut patch $\mathbf{x}(u,v)$. Then every u-parameter curve is a geodesic and a v-parameter curve with $u = u_0$ is a geodesic precisely when $G_u(u_0) = 0$.*

Corollary 6 *For a surface of revolution having parametrization*

$$\mathbf{x}(u,v) = (g(u), h(u)\cos v, h(u)\sin v),$$

any meridian is a geodesic and a parallel is a geodesic precisely when $h'(u_0) = 0$.