4.2 Other Curvatures

Definition 4.2 *The* ***Gaussian curvature*** *of a surface M at $p \in M$ is defined to be $K(p) = \det(S_p)$.*

The ***mean curvature*** *of a surface M at $p \in M$ is defined in terms of the trace $H(p) = \frac{1}{2}\operatorname{trace}(S_p)$.*

It appears that K and H are defined solely in terms of the shape operator. They are but we shall see that we can calculate them by calculus alone. We have seen that the matrix of the shape operator with respect to a basis of principal vectors is given by

$$\begin{bmatrix} k_1(p) & 0 \\ 0 & k_2(p) \end{bmatrix}$$

and the determinant and trace are then k_1k_2 and $k_1 + k_2$. Since the determinant and trace do not change in a change of basis, we have that

$$K = k_1k_2 \quad \text{and} \quad H = \frac{k_1 + k_2}{2}.$$

Now, recall that we had to make a choice in the sign of U. If we had chosen $-U$ as the normal instead of U, then $k(\mathbf{u})$ changes sign. Since the Gaussian curvature is the product of two such changes, K does not change sign by switching the choice of U. However, H does change sign under a change in unit normal.

Suppose that $K(p) > 0$. Since $K = k_1k_2$, then both k_1 and k_2 have the same sign. Since $k_1 = \max k(\mathbf{u})$ and $k_2 = \min k(\mathbf{u})$, then all $k(\mathbf{u})$ have the same sign for all $\mathbf{u}$. So if $k(\mathbf{u}) > 0$ for all $\mathbf{u}$, then M bends toward U in every direction. If $k(\mathbf{u}) < 0$ for all $\mathbf{u}$ then M bends away from U in every direction.

How can we compute the curvature? Let $\mathbf{v}$ and $\mathbf{w}$ be linearly independent vectors in T_pM. Since they are linearly independent and M is a surface, they form a basis for T_pM. Thus,

$$S(\mathbf{v}) = a\mathbf{v} + b\mathbf{w} \quad \text{and} \quad S(\mathbf{w}) = c\mathbf{v} + d\mathbf{w}.$$

The matrix for S with respect to the basis $\{\mathbf{v}, \mathbf{w}\}$ is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

From this we see that $\det S = ad - bc = K$ and $\operatorname{trace} S = a + d = 2H$.

We will need the following calculations

$$\begin{aligned}
S(\mathbf{v}) \times S(\mathbf{w}) &= (a\mathbf{v} + b\mathbf{w}) \times (c\mathbf{v} + d\mathbf{w}) \\
&= ac(\mathbf{v} \times \mathbf{v}) + ad(\mathbf{v} \times \mathbf{w}) + bc(\mathbf{w} \times \mathbf{v}) + cd(\mathbf{w} \times \mathbf{w}) \\
&= 0 + (ad - bc)(\mathbf{v} \times \mathbf{w}) + 0 \\
&= K(\mathbf{v} \times \mathbf{w}) \\
S(\mathbf{v}) \times \mathbf{w} + \mathbf{v} \times S(\mathbf{w}) &= (a\mathbf{v} + b\mathbf{w}) \times \mathbf{w} + \mathbf{v} \times (c\mathbf{v} + d\mathbf{w}) \\
&= a(\mathbf{v} \times \mathbf{w}) + d(\mathbf{v} \times \mathbf{w}) \\
&= \operatorname{trace} S(\mathbf{v} \times \mathbf{w}) \\
&= 2H(\mathbf{v} \times \mathbf{w})
\end{aligned}$$

Lemma 4.3 (Lagrange Identity) *For vectors* $\mathbf{u}$*,* $\mathbf{v}$*,* $\mathbf{w}$*, and* $\mathbf{x}$

$$(\mathbf{v}\times\mathbf{w})\cdot(\mathbf{u}\times\mathbf{x}) = (\mathbf{v}\cdot\mathbf{u})(\mathbf{w}\cdot\mathbf{x}) - (\mathbf{v}\cdot\mathbf{x})(\mathbf{w}\cdot\mathbf{u}).$$

Applying this lemma to the formulas above we get:

$$(S(\mathbf{v})\times S(\mathbf{w}))\cdot(\mathbf{v}\times\mathbf{w}) = K(\mathbf{v}\times\mathbf{w})\cdot(\mathbf{v}\times\mathbf{w})$$
$$(S(\mathbf{v})\cdot\mathbf{v})(S(\mathbf{w})\cdot\mathbf{w}) - (S(\mathbf{v})\cdot\mathbf{w})(S(\mathbf{w})\cdot\mathbf{v}) = K((\mathbf{v}\cdot\mathbf{v})(\mathbf{w}\cdot\mathbf{w}) - (\mathbf{v}\cdot\mathbf{w})(\mathbf{w}\cdot\mathbf{v}))$$

solving for K, we get

$$K = \frac{(S(\mathbf{v})\cdot\mathbf{v})(S(\mathbf{w})\cdot\mathbf{w}) - (S(\mathbf{v})\cdot\mathbf{w})(S(\mathbf{w})\cdot\mathbf{v})}{(\mathbf{v}\cdot\mathbf{v})(\mathbf{w}\cdot\mathbf{w}) - (\mathbf{v}\cdot\mathbf{w})(\mathbf{w}\cdot\mathbf{v})}$$

and

$$(S(\mathbf{v})\times\mathbf{w} + \mathbf{v}\times S(\mathbf{w}))\cdot(\mathbf{v}\times\mathbf{w}) = 2H(\mathbf{v}\times\mathbf{w})\cdot(\mathbf{v}\times\mathbf{w})$$
$$\begin{aligned}&(S(\mathbf{v})\cdot\mathbf{v})(\mathbf{w}\cdot\mathbf{w}) - (S(\mathbf{v})\cdot\mathbf{w})(\mathbf{w}\cdot\mathbf{v}) + \\ &\quad(\mathbf{v}\cdot\mathbf{v})(S(\mathbf{w})\cdot\mathbf{w}) - (\mathbf{v}\cdot\mathbf{w})(S(\mathbf{w})\cdot\mathbf{v}) = 2H((\mathbf{v}\cdot\mathbf{v})(\mathbf{w}\cdot\mathbf{w}) - (\mathbf{v}\cdot\mathbf{w})(\mathbf{w}\cdot\mathbf{v}))\end{aligned}$$

Dividing we obtain

$$H = \frac{(S(\mathbf{v})\cdot\mathbf{v})(\mathbf{w}\cdot\mathbf{w}) - (S(\mathbf{v})\cdot\mathbf{w})(\mathbf{w}\cdot\mathbf{v}) + (\mathbf{v}\cdot\mathbf{v})(S(\mathbf{w})\cdot\mathbf{w}) - (\mathbf{v}\cdot\mathbf{w})(S(\mathbf{w})\cdot\mathbf{v})}{2((\mathbf{v}\cdot\mathbf{v})(\mathbf{w}\cdot\mathbf{w}) - (\mathbf{v}\cdot\mathbf{w})(\mathbf{w}\cdot\mathbf{v}))}$$

Now, let's look at how the principal curvatures are determined by the Gaussian and mean curvatures. We know that $K = k_1k_2$ and $2H = k_1 + k_2$. We want to find k_i in terms of K and H. One interesting way to see this is that k_1 and k_2 are eigenvalues for the shape operator, S. In terms of the principal directions, the matrix for S is $\begin{bmatrix} a & b \\ c & d\end{bmatrix}$. The eigenvalues are the roots of the characteristic polynomial for this matrix. The characteristic polynomial for this matrix is

$$\begin{aligned} p(x) &= x^2 - (a+d)x + (ad - bc) \\ &= x^2 - \operatorname{trace} Sx + \det S \\ &= x^2 - 2Hx + K \end{aligned}$$

The roots are $k_1 = H + \sqrt{H^2 - K}$ and $k_2 = H - \sqrt{H^2 - K}$.

What is the geometry of the Gaussian curvature? Let's go back and look at the Gauss map. The derivative of the Gauss map, $G\colon M \to S^2$, is the negative of the shape operator: $G_*(\mathbf{v}) = -S(\mathbf{v})$. Then for the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ we have

$$\begin{aligned} G_*(\mathbf{x}_u)\times G_*(\mathbf{x}_v) &= -S(\mathbf{x}_u)\times(-S(\mathbf{x}_v)) \\ &= K\mathbf{x}_u\times\mathbf{x}_v \end{aligned}$$

Geometrically, $|G_*(\mathbf{x}_u)\times G_*(\mathbf{x}_v)|$ and $|\mathbf{x}_u\times\mathbf{x}_v|$ represent areas — infinitessimal areas on the image of the Gauss map on S^2 and M, respectively. Our above formula says that the ratio of these two infinitessimal areas is $|K|$. Another way to say this is that for any small open neighborhood V containing $p \in M$ then

$$|K| = \lim_{V\to p}\frac{\text{Area } G(V)}{\text{Area } V}.$$

This says that the magnitude of the Gaussian curvature measures the way in which the unit normal expands or contracts area. A surface M is said to be *flat* if $K(p) = 0$ for every $p \in M$ and it is said to be *minimal* if $H(p) = 0$ for every $p \in M$.