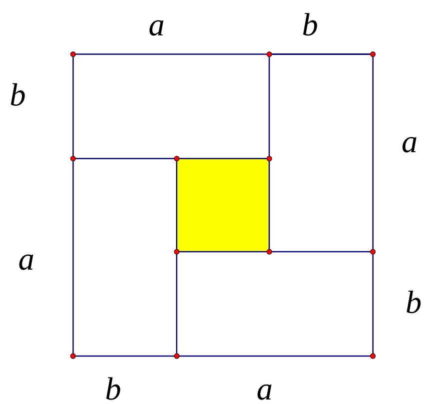
MATH 6118-090
Non-euclidean Geometry

Exercise Set #1

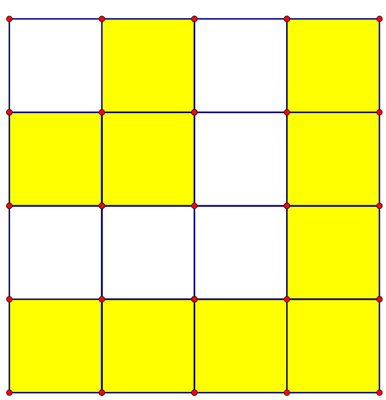
1. (Page 9, #4) To the naked eye, it appears that the area of a circle may be halfway between those of an inscribed square and a circumscribed square. If this observation is taken as true and accurate, what value must π take?

2. (Page 10, #5) Pythagorean triples are positive integers, a, b, and c that satisfy the Pythagorean equation $a^2 + b^2 = c^2$.
   a. Find three sets of Pythagorean triples other than 3,4,5.
   b. Prove that 3,4,5, is the only Pythagorean triple with consecutive integers as sides.
   c. Prove that no isosceles right triangles exist for which the sides are Pythagorean triples.
   d. Prove that in every Pythagorean triple, at least one leg is a multiple of three.

3. (Page 10, #9) Based on the following geometric model, prove that $(a+b)^2 > (a-b)^2$.

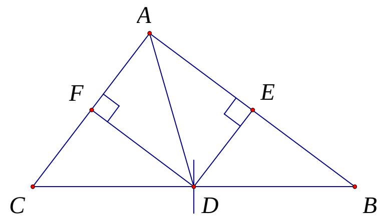


4. (Page 10, #10) Use the following geometric model to find a formula for the sum of the first n odd counting numbers. Justify your answer.

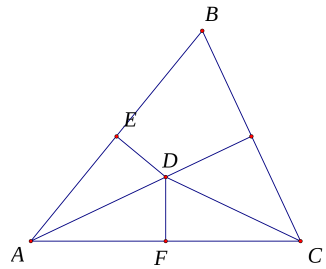


5. What follows is a well-known and involved argument that pretends to prove that all triangles are isosceles. Find the flaw in the argument.

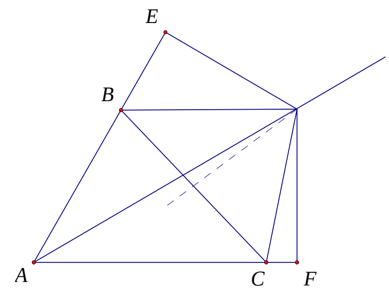
Given $\triangle ABC$. Construct the bisector of $\sphericalangle A$ and the perpendicular bisector of the side BC opposite to $\sphericalangle A$. Consider the following cases.



Case 3



Case 2



Case 4

Case 1. The bisector of $\sphericalangle A$ and the perpendicular bisector of segment BC are either parallel or identical. In either case, the bisector of $\sphericalangle A$ is perpendicular to BC and hence, by definition, is an altitude. Therefore, the triangle is isosceles. (The conclusion follows from the Euclidean theorem: if an angle bisector and altitude from the same vertex of a triangle coincide, the triangle is isosceles.)

Suppose now that the bisector of $\sphericalangle A$ and the perpendicular bisector of the side opposite are not parallel and do not coincide. Then they intersect in exactly one point, D, and there are three cases to consider.

Case 2: The point D is inside the triangle.

Case 3: The point D is on the triangle.

Case 4: The point D is outside the triangle.

For each case construct DF perpendicular to AB and DF perpendicular to AC, and for cases 2 and 4 join D to B and D to C. In each case, the following proof now holds:

$DE \cong DF$ because all points on an angle bisector are equidistant from the sides of the angle; $DA \cong DA$, and $\sphericalangle DEA$ and $\sphericalangle DFA$ are right angles; hence, $\triangle ADE$ is congruent to $\triangle ADF$ by the hypotenuse-leg theorem of Euclidean geometry. (You could also use SAA). Therefore we have $AE \cong AF$, and $\sphericalangle DEB$ and $\sphericalangle DFC$ are right angles. Hence, $\triangle DEB$ is congruent to $\triangle DFC$ by the hypotenuse-leg theorem, and hence $FC \cong BE$. It now follows that $AB \cong AC$ — in cases 2 and 3 by addition and in case 4 by subtraction. The triangle is therefore isosceles.