Chapter 12

Integrals

12.1 Path Intgrals

Let $\alpha\colon [a,b] \to \mathbb{R}^2$ or $\mathbb{R}^3$ be a C^1 path. The function $\phi\colon [c,d] \to [a,b]$ is called *bijective* if and only if it is one-to-one and onto. The composition $\beta = \alpha \circ \phi\colon [c,d] \to \mathbb{R}^2$ (or $\mathbb{R}^3$) is called a *reparametrization* of α. We saw earlier that

$$\beta'(t) = \alpha'(\phi(t))\phi'(t).$$

Thus,

$$\|\beta'(t)\| = |\phi'(t)|\|\alpha'(\phi(t))\|$$

so the reparametrization is a change of speed and/or direction. Now, $\phi\colon [c,d] \to [a,b]$ must map the endpoints to endpoints. Thus, either $\phi(c) = a$ and $\phi(d) = b$ or $\phi(c) = b$ and $\phi(d) = a$. In the former case, $\beta(c) = \alpha(\phi(c)) = \alpha(a)$ and $\beta(d) = \alpha(\phi(d)) = \alpha(b)$ and the reparametrization is called *orientation preserving*. The other case is called *orientation reversing*.

Theorem 12.1 *A reparametrization is orientation preserving if and only if $\phi' > 0$ and orientation reversing if and only if $\phi' < 0$.*

Definition 12.1 *The image of a one-to-one, piecewise C^1 path $\alpha\colon [a,b] \to \mathbb{R}^2$ (or $\mathbb{R}^3$) is called a* simple curve.

A simple curve means that no two points of the interval $[a,b]$ are mapped onto the same point on the curve, a simple curve cannot intersect itself.

Definition 12.2 *The image of a piecewise C^1 path $\alpha\colon [a,b] \to \mathbb{R}^2$ (or $\mathbb{R}^3$) that is one-to-one on $[a,b)$ and is such that $\alpha(a) = \alpha(b)$ is called a* simple closed curve.

12.2 Path Integrals of Real-Valued Functions

We used the Riemann sum to define the definite integral of a real-valued function on an interval $[a, b]$. We will use this same technique to define the integral of a real-valued function over a path in $\mathbb{R}^n$.

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a continuous function and let $\sigma\colon [a, b] \to \mathbb{R}^n$ be a path in $\mathbb{R}^n$. The composition $f(\sigma(t))$ represents the values of the function f along the curve σ. Let P be a partition of $[a, b]$, say $P = \{t_0 = a, t_1, \ldots, t_n = b\}$. We can approximate the path σ by the polygonal path p_n, whose vertices are $\{\sigma(a) = \sigma(t_0), \sigma(t_1), \ldots, \sigma(t_n) = \sigma(b)\}$. We can approximate the length of σ_i connecting $\sigma(t_i)$ and $\sigma(t_{i+1})$ as $\|\sigma'(t_i)\|\Delta t_i$, where $\Delta t_i = t_{i+1} - t_i$. Now we form the sums:

$$A_n = \sum_{i=0}^{n-1} f(\sigma(t_i))\|\sigma'(t_i)\|\Delta t_i.$$

If f were positive, then A_n could represent the approximate area of the "fence" built along σ from $\sigma(a)$ to $\sigma(b)$ whose height is determined by f.

Definition 12.3 *Let $\sigma\colon [a, b] \to \mathbb{R}^n$ be a C^1 path and let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a function such that the composition $f(\sigma(t))$ is continuous on $[a, b]$. The* path integral $\int_\sigma f\, ds$ *of f along σ is given by*

$$\int_\sigma f\, ds = \int_a^b f(\sigma(t)\|\sigma'(t)\|dt = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(\sigma(t_i))\|\sigma'(t_i)\|\Delta t_i,$$

if this limit exists.

Recall that

$$\|\sigma'(t)\| = \sqrt{\sum_{i=1}^{n} (x_i'(t))^2}.$$

Example 12.1 Let $f(x, y, z) = xyz$ and $\sigma(t) = (-\sin t, \sqrt{2}\cos t, \sin t)$, $t \in [0, \pi/2]$. Find $\int_\sigma f\, ds$.

First,

$$\begin{aligned}
f(\sigma(t)) &= f(-\sin t, \sqrt{2}\cos t, \sin t) = -\sqrt{2}\sin^2 t\cos t \\
\sigma'(t) &= (-\cos t, -\sqrt{2}\sin t, \cos t) \\
\|\sigma'(t)\| &= \sqrt{2} \\
\int_\sigma f\, ds &= -\sqrt{2}\int_0^{\pi/2} (\sin^2 t\cos t)\sqrt{2}dt = -\frac{2}{3}\sin^3 t\Big|_0^{\pi/2} = -\frac{2}{3}.
\end{aligned}$$

Example 12.2 Let $f(x, y) = 2x + y$. Consider the path integral $\int_\sigma f\,ds$ along the following paths in $\mathbb{R}^2$ joining the points $(1, 0)$ and $(0, 1)$:

1. Counterclockwise along the quarter circle $\sigma_1(t) = (\cos t, \sin t)$, $t \in [0, \pi/2]$.
2. Along the straight-line segment $\sigma_2(t) = (1, 0) + t(-1, 1) = (1 - t, t)$, $t \in [0, 1]$.
3. Along the piecewise C^1 path σ_3 that consists of the path $\sigma_4(t) = (1 - t, 0)$, $t \in [0, 1]$, followed by $\sigma_5(t) = (0, t)$, $t \in [0, 1]$.

1. We have $\sigma_1'(t) = (-\sin t, \cos t)$ and $\|\sigma_1'(t)\| = 1$, and

$$\int_{\sigma_1} f\,ds = \int_0^1 (2\cos t + \sin t)dt = 3.$$

2. Here we have $\sigma_2'(t) = (-1, 1)$ and $\|\sigma_2(t)\| = \sqrt{2}$, and

$$\int_{\sigma_2} f\,ds = \int_0^1 (2 - t)\sqrt{2}dt = \frac{3\sqrt{2}}{2}.$$

3. Here $\|\sigma_4(t)\| = \|\sigma_5(t)\| = 1$ and

$$\int_{\sigma_3} f\,ds = \int_{\sigma_4} f\,ds + \int_{\sigma_5} f\,ds = \int_0^1 (2 - 2t)dt + \int_0^1 t\,dt = \frac{3}{2}.$$

Theorem 12.2 *Let σ be a C^1 path in $\mathbb{R}^n$, let f be a real-valued function continuous on the image of σ, and let $\tau = \sigma \circ \phi$ be a reparametrization of σ. Then*

$$\int_\sigma f\,ds = \int_\tau f\,ds.$$

Note that this says that the integral is independent of any reparametrization.

12.3 Path Integrals of Vector Fields

Let $\mathbf{F}$ be a vector field defined on the area that contains the curve σ. Again, let P be a partition of $[a, b]$ and let $\Delta\sigma_i$ be the vector that runs from $\sigma(t_i)$ to $\sigma(t_{i+1})$. Then $\Delta\sigma_i$ is approximately equal to $\sigma'(t_i)\Delta t_i$. Then for each $i = 0, \ldots, n-1$, we can form the dot product $\mathbf{F}(\sigma(t_i)) \cdot \Delta\sigma_i$. We can then form the Riemann sum

$$\sum_{i=0}^{n-1} \mathbf{F}(\sigma(t_i)) \cdot \Delta\sigma_i.$$

If this limit exists, then it is called the path integral (or line integral) of $\mathbf{F}$ along σ and is denoted by $\int_\sigma \mathbf{F} \cdot d\mathbf{s}$.

Definition 12.4 *Let $\sigma\colon [a,b] \to \mathbb{R}^n$ be a C^1 path, and let $\mathbf{F}\colon \mathbb{R}^n \to \mathbb{R}$ be a vector field such that the composition $\mathbf{F}(\sigma(t))$ is continuous on $[a,b]$. The* path integral $\int_\sigma \mathbf{F}\cdot d\mathbf{s}$ *of $\mathbf{F}$ along σ is given by*

$$\int_\sigma \mathbf{F}\cdot d\mathbf{s} = \int_a^b \mathbf{F}(\sigma(t))\cdot \sigma'(t)\,dt.$$

In the case where $\mathbf{F}$ represents a force, the path integral gives the work done by $\mathbf{F}$ along σ.

Theorem 12.3 *Let $\mathbf{F}$ be a continuous vector field on $\mathbb{R}^n$, let $\sigma\colon [a,b] \to \mathbb{R}^n$ be a C^1 curve, and let $\tau(t) = \sigma(\phi(t))$ be a reparametrization of σ, where $\phi\colon [c,d] \to [a,b]$. Then*

$$\int_\sigma \mathbf{F}\cdot d\mathbf{s} = \begin{cases} \int_\tau \mathbf{F}\cdot d\mathbf{s} & \text{if } \phi \text{ is orientation preserving} \\ -\int_\tau \mathbf{F}\cdot d\mathbf{s} & \text{if } \phi \text{ is orientation reversing} \end{cases}$$

Definition 12.5 *The line integral $\int_\sigma \mathbf{F}\cdot d\mathbf{s}$ of a continuous vector field $\mathbf{F}$ around a oriented simple closed curve σ is called the* circulation of $\mathbf{F}$ around σ.

12.4 Double Integrals

As we did for a function from $\mathbb{R} \to \mathbb{R}$ we can also talk about the integral of a function $\mathbb{R}^2 \to \mathbb{R}$. Here we have two different dimensions along which to partition.

Divide the intervals $[a,b]$ and $[c,d]$ into n subintervals

$$P = \{a = x_1, x_2, \ldots, x_{n+1} = b\} \text{ and } Q = \{c = y_1, y_2, \ldots, y_{n+1} = d\}$$

and form the rectangles $R_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, $i, j = 1, \ldots, n$. This divides the rectangle $R = [a,b] \times [c,d]$ into n^2 subrectangles R_{ij}. The sides of R_{ij} are $\Delta x_i = x_{i+1} - x_i$ and $\Delta y_j = y_{j+1} - y_j$ and its area is $\Delta A_{ij} = \Delta x_i \Delta y_j$.

Choose a point (x_i^*, y_j^*) in each R_{ij} and build a parallelpiped over R_{ij} whose height is equal to the value $f(x_i^*, y_j^*)$ of f at (x_i^*, y_j^*). The volume of the parallelpiped, $f(x_i^*, y_j^*)\Delta A_{ij}$ approximates the volume of the three-dimensional region under the surface $z = f(x,y)$ and above R_{ij}. The sum of the volumes of all $n62$ parallepipeds is called the double Riemann sum and is

$$\mathscr{R}_n = \sum_{i=1}^n \sum_{j=1}^n f(x_i^*, y_j^*)\Delta A_{ij}$$

approximates the volume of the region under the surface $z = f(x,y)$ over the rectangle $[a,b] \times [c,d]$.

A function $f(x, y)$ defined on a rectangle $R \subseteq \mathbb{R}^2$ is called *integrable on* R if the limit of the sequence of Riemann sums

$$\mathscr{R}_n = \sum_{i=1}^{n} \sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta A_{ij}$$

as $n \to \infty$ exists and does not depend on the way that points are chosen in each subrectangle R_{ij}.

If f is integrable, then the *double integral* $\iint_R f\,dA$ *of* f *over the rectangle* R is given by

$$\iint_R f\,dA = \lim_{n\to\infty} \mathscr{R}_n = \lim n \to \infty \sum_{i=1}^{n} \sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta A_{ij}.$$

Theorem 12.4 (Properties of Double Integrals) *Let f and g be integrable functions defined on a rectangle R and let c be a constant. Then*

1. *The function $f + g$ is integrable, and*

$$\iint_R (f+g)\,dA = \iint_R f\,dA + \iint_R g\,dA.$$

2. *The function cf is integrable, and*

$$\iint_R cf\,dA = c\iint_R f\,dA.$$

3. *If R is divided into n rectangles R_i, $(i = 1, \ldots, n)$ that are mutually disjoint, then f is integrable over each R_i and*

$$\iint_R f\,dA = \sum_{i=1}^{n} \iint_{R_i} f\,dA.$$

4. *$f(x, y) \le g(x, y)$ on R, then*

$$\iint_R f\,dA \le \iint_R g\,dA.$$

5. *The absolute value of the double integral satisfies*

$$\left| \iint_R f\,dA \right| \le \iint_R |f|\,dA.$$

Theorem 12.5 *Let $h(x, y)$ be a function of two variables, let $R = [a, b] \times [c, d]$, and let $f, g \colon \mathbb{R} \to \mathbb{R}$ be so that*

$$h(x, y) = f(x)g(y).$$

Then

$$\iint_R h(x, y)\,dA = \int_a^b f(x)\,dx \int_c^d g(y)\,dy.$$

Theorem 12.6 (Fubini's Theorem) *If $f = f(x, y)$ is a continuous function defined on a rectangle $R = [a, b] \times [c, d]$ in $\mathbb{R}^2$, then*

$$\iint_R f\,dA = \int_a^b \left(\int_c^d f(x, y)\,dy \right) dx = \int_c^d \left(\int_a^b f(x, y)\,dx \right) dy.$$

12.5 Triple Integrals

We define the integral of a function $f\colon \mathbb{R}^3 \to \mathbb{R}$ similarly. If $W = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$, then we can form the *triple Riemann sum*

$$\mathscr{R}_n = \sum_i \sum_j \sum_k f(x_i^*, y_j^*, z_k^*)\Delta x_i \Delta y_j \Delta z_k.$$

The *triple integral* $\iiint_W f\,dV$ *of* f *over* W is defined by

$$\iiint_W f\,dV = \lim_{n\to\infty} \mathscr{R}_n$$

when this limit exists.

Theorem 12.7 (Fubini's Theorem) *If $f = f(x, y, z)$ is a continuous function defined on a rectangular box $W = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ in $\mathbb{R}^3$, then*

$$\begin{aligned}
\iint_R f\,dA &= \int_{a_3}^{b_3} \left(\int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} f(x, y, z)\,dx \right) dy \right) dz \\
&= \int_{a_1}^{b_1} \left(\int_{a_3}^{b_3} \left(\int_{a_2}^{b_2} f(x, y, z)\,dy \right) dz \right) dx \\
&= \int_{a_3}^{b_3} \left(\int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} f(x, y, z)\,dy \right) dx \right) dz
\end{aligned}$$

etc. There are six interated integrals that are all equal.

12.6 Change of Variables

The change of variables is much like our Chain Rule for functions of one variable. This section is mostly definitions, just so we can see how things work.

Assume that $f =\colon \mathbb{R}^n \to \mathbb{R}$ is a function of n variables. We will consider $n = 2$ and $n = 3$ for the most part, but everything is the same for arbitrary n. A *change of variables* is a function $T\colon \mathbb{R}^n \to \mathbb{R}^n$. We think of it as setting $x_i = x_i(u_1, \dots, u_n)$, $i = 1, \dots, n$, where x_i is a differentiable function of the u_j. For $n = 2$ this looks like

$$T(u, v) = (x(u, v), y(u, v)).$$

Standard examples are polar coordinates for $n = 2$ and cylindrical and spherical coordinates for $n = 3$.

Let $T(u, v) = (x(u, v), y(u, v))$ be a C^1 function. The determinant of the derivative DT of T is called the *Jacobian* of T and is denoted by $J(x, y; u, v)$ or $\partial(x, y)/\partial(u, v)$. Hence

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$$

Thus, for $T\colon \mathbb{R}^3 \to \mathbb{R}^3$

$$J(x,y,z;u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix}$$

and for $T\colon \mathbb{R}^n \to \mathbb{R}^n$

$$J(x_1,\dots,x_n;u_1,\dots,u_n) = \frac{\partial(x_1,\dots,x_n)}{\partial(u_1,\dots,u_n)} = \left|\frac{\partial x_i}{\partial u_j}\right| = \begin{vmatrix} \partial x_1/\partial u_1 & \partial x_1/\partial u_2 & \dots & \partial x_1/\partial u_n \\ \partial x_2/\partial u_1 & \partial x_2/\partial u_2 & \dots & \partial x_2/\partial u_n \\ \vdots & \vdots & & \vdots \\ \partial x_n/\partial u_1 & \partial x_n/\partial u_2 & \dots & \partial x_n/\partial u_n \end{vmatrix}$$

Example 12.3 For polar coordinates, $x = r\cos\theta$ and $y = r\sin\theta$ we have

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

For cylindrical coordinates, $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$, so

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

For spherical coordinates, $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$, and $z = \rho\cos\phi$, and

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix} = \rho^2\sin\phi.$$

Theorem 12.8 (Change of Variables Formula, $n = 2$**)** *Let D and D^* be elementary regions in $\mathbb{R}^2$ and let $T\colon D^* \to D$ be a C^1, one-to-one map such that $T(D^*) = D$. For any integrable function $f\colon D \to \mathbb{R}$*

$$\iint_D f(x,y)dA = \iint_{D^*} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA^*,$$

where $\dfrac{\partial(x,y)}{\partial(u,v)}$ is the Jacobian of T.

Thus, for a change to polar coordinates

$$\iint_D f(x,y)dxdy = \iint_{D^*} f(r,\theta)r\,drd\theta.$$

We can see what the Change of Variables is for general n.

Theorem 12.9 (General Change of Variables Formula) *Let R and R^* be elementary regions in $\mathbb{R}^n$ and let $T\colon D^* \to D$ be a C^1, one-to-one map such that $T(D^*) = D$. For any integrable function $f\colon D \to \mathbb{R}$*

$$\int \cdots \int_D f(x_1,\ldots,x_n)dW = \int \cdots \int_{D^*} f(x_1(u_1,\ldots,u_n),\ldots,x_n(u_1,\ldots,u_n)) \left|\frac{\partial(x_1,\ldots,x_n)}{\partial(u_1,\ldots,u_n)}\right| dW^*,$$

where $\dfrac{\partial(x_1,\ldots,x_n)}{\partial(u_1,\ldots,u_n)}$ *is the Jacobian of T.*

This intimates that for a change to cylindrical coordinates,

$$\iiint_D f(x,y,z)dxdydz = \iiint_{D^*} f(r,\theta,z)r\,drd\theta dz,$$

and for a change to spherical coordinates,

$$\iiint_D f(x,y,z)dxdydz = \iiint_{D^*} f(\rho,\theta,\phi)\rho^2\sin\phi\,d\rho d\theta d\phi.$$