# *Problems for the Final Exam*
# MATH 6102 - Spring 2009

1. If $f(1) = 1$ and $f'(1) = 2$, compute
$$\lim_{x\to 1} \frac{[f(x)]^2 - 1}{x^2 - 1}.$$

2. If $a, b, c, d \in \mathbb{R}$ find
$$\lim_{x\to 0} \frac{\sin ax \sin bx}{\sin cx \sin dx}.$$

3. Suppose that $f$ is differentiable with derivative $f'(x) = (1+x^3)^{-1/2}$. Show that $g = f^{-1}$ satisfies $g''(x) = \frac{3}{2}[g(x)]^2$. **DO NOT FIND $f(x)$.**

4. Suppose that $f$ and $g$ are two differentiable functions which satisfy $fg' - f'g = 0$. Prove that if $a$ and $b$ are adjacent zeros of $f$[1], and $g(a)$ and $g(b)$ are not both 0, then $g(x) = 0$ for some $x$ between $a$ and $b$.
Hint: Derive a contradiction from the assumption that $g(x) \neq 0$ for all $x$ between $a$ and $b$.

5. Find $f'$ in terms of $g'$ if
   (a) $f(x) = g(x + g(a))$.
   (b) $f(x) = g(x \cdot g(a))$
   (c) $f(x) = g(x + g(x))$
   (d) $f(x) = g(x)(x - a)$
   (e) $f(x) = g(a)(x - a)$
   (f) $f(x + 3) = g(x^2)$

6. Find a function $g$ such that
   (a) $\int_0^x tg(t)\,dt = x + x^2$
   (b) $\int_0^{x^2} tg(t)\,dt = x + x^2$

   (Note that $g$ is not assumed to be continuous at $x = 0$.)

7. Find $F'(x)$ if $F(x) = \int_0^x xf(t)\,dt$. (The answer is *not* $xf(x)$; you should perform an obvious manipulation on the integral before trying to find $F'$.)

8. Calculate $\lim_{x\to 0} \frac{1}{x} \int_0^x e^{t^2}\,dt$.   Hint: Think l'Hospital.

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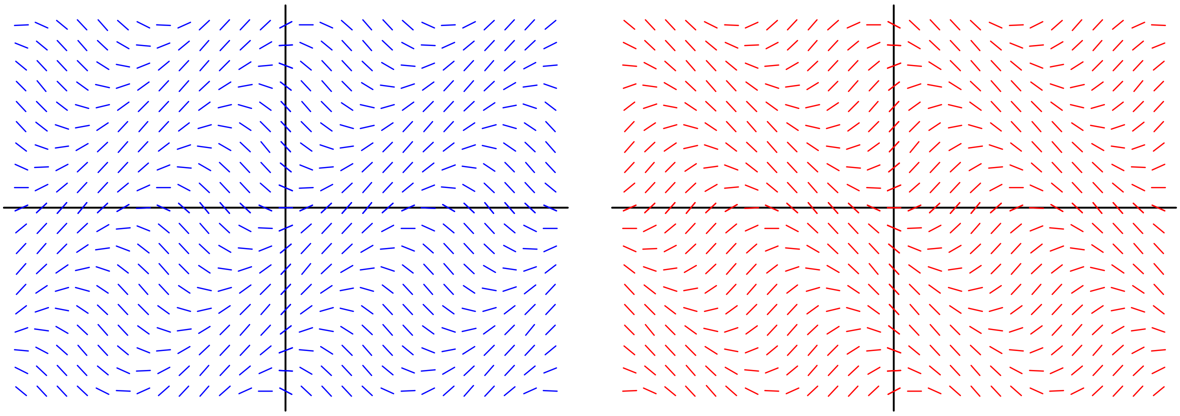
[1] This means that $f(a) = f(b) = 0$ and $f(x) \neq 0$ for all $x \in (a, b)$.

9. One of the following slope fields is for the differential equation $dy/dx = \sin(x + y)$ and the other is for the differential equation $dy/dx = \sin(x - y)$.



   (a) Where would you expect the slope to be zero in the slope field for $dy/dx = \sin(x + y)$? Where would you expect the slope to be zero in the slope field for $dy/dx = \sin(x - y)$?
   (b) Which slope field goes with which equation?
   (c) One of the graphs shows "stripes" with slope 1, the other shows "stripes" with slope $-1$. Which is which and why?
   (d) Can you determine where $\pi$ and $-\pi$ would be on the $x$-axis in both graphs?

10. Find the solution to the differential equation

$$\frac{dy}{dx} = \frac{x^2 \sin(x^3)}{2y}$$

that passes through the point $(2, 7)$.

11. Consider the differential equation

$$\frac{dy}{dx} = x^2 - y^2 + 1.$$

Assume that the solution curve passes through the point $(0, 1)$. Use Euler's method to approximate $y(1)$ by using 5 steps.

12. Use Picard's Method to find the first four nonzero terms in the approximation to the solution of the initial value problem:

$$\frac{dy}{dx} = x^2 y + \frac{x^5}{3} \text{ with } y(0) = 0.$$