MATH 6102 — SPRING 2009
ASSIGNMENT 1

January 26, 2009
Due February 2, 2009

Evaluate the following limits.

1. $\lim_{x\to 2}(x^2 - 4x)$

$$\lim_{x\to 2}(x^2 - 4x) = -4.$$

2. $\lim_{x\to 0}\dfrac{3^x - 3^{-x}}{3^x + 3^{-x}}$

$$\lim_{x\to 0}\frac{3^x - 3^{-x}}{3^x + 3^{-x}} = \frac{0}{2} = 0.$$

3. $\lim_{x\to 2}\dfrac{x^2 - 4}{x^2 - 5x + 6}$

$$\lim_{x\to 2}\frac{x^2 - 4}{x^2 - 5x + 6} = \lim_{x\to 2}\frac{x+2}{x-3} = -4.$$

4. $\lim_{x\to -1}\dfrac{x^2 + 3x + 2}{x^2 + 4x + 3}$

$$\lim_{x\to -1}\frac{x^2 + 3x + 2}{x^2 + 4x + 3} = \lim_{x\to -1}\frac{x+2}{x+3} = \frac{1}{2}.$$

5. $\lim_{x\to 2}\dfrac{x - 2}{\sqrt{x^2 - 4}}$

$$\lim_{x\to 2}\frac{x - 2}{\sqrt{x^2 - 4}} = \lim_{x\to 2}\frac{\sqrt{x-2}\sqrt{x-2}}{\sqrt{x+2}\sqrt{x-2}} = 0.$$

6. $\lim_{x\to \infty}\dfrac{7x^9 - 4x^4 + 2x - 13}{-3x^9 + x^8 - 5x^2 + 2x}$

$$\lim_{x\to \infty}\frac{7x^9 - 4x^4 + 2x - 13}{-3x^9 + x^8 - 5x^2 + 2x} = -\frac{7}{3}.$$

7. $\lim_{x\to \infty}\dfrac{3^x - 3^{-x}}{3^x + 3^{-x}}$

$$\lim_{x\to \infty}\frac{3^x - 3^{-x}}{3^x + 3^{-x}} = 1.$$

8. Show that $\lim_{x\to\infty} x - \sqrt{x^2 - 1} = 0$.

$$\begin{aligned}\lim_{x\to\infty} x - \sqrt{x^2-1} &= \lim_{x\to\infty}\frac{x + \sqrt{x^2-1}}{x + \sqrt{x^2-1}}(x - \sqrt{x^2-1}) \\ &= \lim_{x\to\infty}\frac{x^2 - x^2 + 1}{x + \sqrt{x^2-1}} = 0\end{aligned}$$

9. *Show that the hyperbola* $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ *gets arbitrarily close to the asymptote* $y = \dfrac{b}{a}x$ *as* x *approaches* ∞.

   This means that we want to show that the limit of one arm of the parabola is this line. We solve for y first:
   $$y^2 = \frac{b^2}{a^2}x^2 + b^2 \longrightarrow y = b\sqrt{\frac{x^2}{a^2} - 1}.$$
   Thus,
   $$\lim_{x\to\infty} y - \frac{b}{a}x = \lim_{x\to\infty} \frac{b}{a}\sqrt{x^2 - a^2} - \frac{b}{a}x = \frac{b}{a}\lim_{x\to\infty}\left(\sqrt{x^2 - a^2} - x\right) = 0.$$
   This last limit follows from above. Thus, the hyperbola approaches the line as x goes to infinity.

10. *Is it possible that* $\lim_{x\to a}[f(x) + g(x)]$ *might exist even though neither* $\lim_{x\to a} f(x)$ *nor* $\lim_{x\to a} g(x)$ *exists? If it is true, then find an example of two such functions.*

    Let $f(x)$ be the Dirichlet function. Then this has no limit at any real number. Let $g(x) = -f(x)$. Then g has no limit at any real number. However, $f(x) + g(x) = 0$ for all x and it has a limit at every real number.

11. *Is it possible that* $\lim_{x\to a}[f(x)g(x)]$ *might exist even though neither* $\lim_{x\to a} f(x)$ *nor* $\lim_{x\to a} g(x)$ *exists? If it is true, then find an example of two such functions.*

    Let $f(x)$ be the Dirichlet function. Then this has no limit at any real number. Let $g(x) = 1 - f(x)$. Then g has no limit at any real number. However, $f(x) \cdot g(x) = 0$ for all x and it has a limit at every real number.

12. *In the figure below, the circle* C *is fixed and has equation* $(x-1)^2 + y^2 = 1$. *The circle* G *is centered at the origin, has radius* r, *and is shrinking to the origin. The point* P *is the point* $(0, r)$, Q *is the upper point of intersection of the two circles and* R *is the point where the line* PQ *intersects the* x*-axis. What happens to the point* R *as the circle* G *shrinks, that is, what is* $\lim_{r\to 0} R$*?*

    In short, $\lim_{r\to 0} R = (4, 0)$. Now, the question is why.

    First, consider the Geometer's Sketchpad file to see that this is where the point R should go. Why does is stop there? There are a couple of ways to see this. One would be to show that the length of the segment PR goes to 4 as $r \to 0$. A second way would be to find the point of intersection of the ray PQ with the x-axis and find its limit - analytically. Let's try this approach.

    First, the equation of circle G is $x^2 + (y - r)^2 = r^2$. We find when C and G intersect by setting the equations "equal" and solving for one of the variables. Using that variable, we can find the other coordinate of the point of intersection. We will use this point of intersection with the point P to determine the slope of the ray PQ, and using that, we will find the point of intersection of the ray PQ with the x-axis.

How do we set two equations "equal" to one another? Let's find the x-coordinate of one circle and plug that into the second circle.

$$\begin{aligned}(x-1)^2+y^2&=1\\ y&=\sqrt{2x-x^2}\\ x^2+y^2&=r^2\\ x^2+2x-x^2&=r^2\\ x&=\frac{r^2}{2}\end{aligned}$$

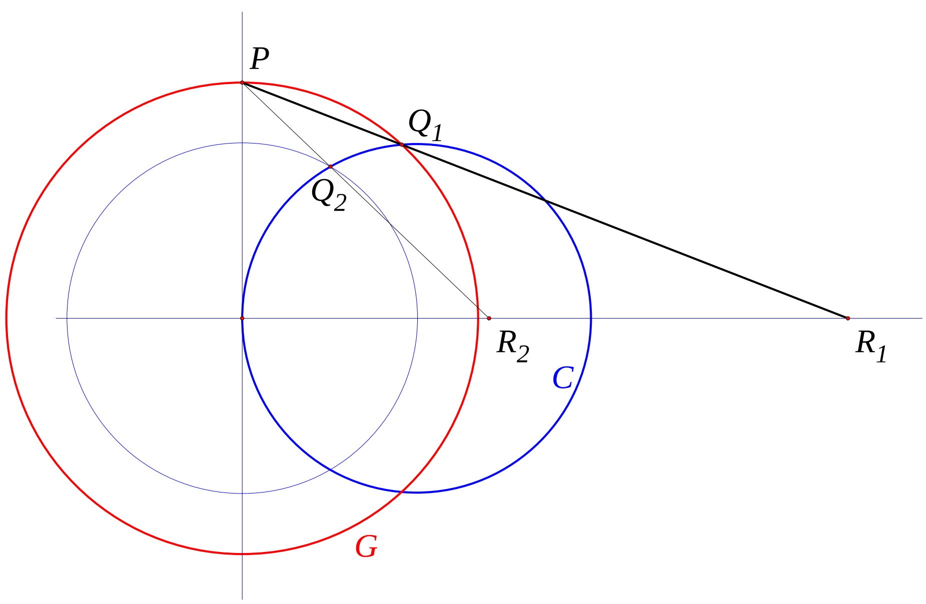
So,

$$y=\frac{r\sqrt{4-r^2}}{2}.$$

$$Q=\left(\frac{r^2}{2},\frac{r\sqrt{4-r^2}}{2}\right).$$

The line joining $(0,r)$ with Q has slope

$$m=\frac{r-\frac{r\sqrt{4-r^2}}{2}}{0-\frac{r^2}{2}}=\frac{-2+\sqrt{4-r^2}}{r}.$$



The equation of this line is then

$$y = \frac{-2 + \sqrt{4 - r^2}}{r} x + r.$$

The x-intercept is

$$x = -\frac{r^2}{-2 + \sqrt{-r^2 + 4}}.$$

Now,

$$\begin{aligned}
\lim_{r \to 0} x &= \lim_{r \to 0} \left(-\frac{r^2}{-2 + \sqrt{-r^2 + 4}} \right) \\
&= \lim_{r \to 0} \left(-\frac{r^2}{-2 + \sqrt{-r^2 + 4}} \frac{2 + \sqrt{-r^2 + 4}}{2 + \sqrt{-r^2 + 4}} \right) \\
&= \lim_{r \to 0} \frac{2r^2 + r^2\sqrt{4 - r^2}}{r^2} \\
&= \lim_{r \to 0} (2 + \sqrt{4 - r^2}) = 4
\end{aligned}$$