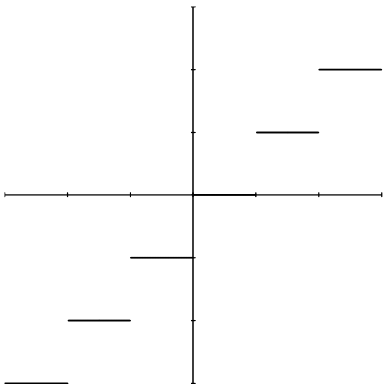
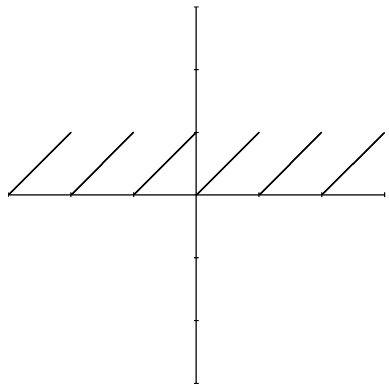
MATH 6102 — SPRING 2009
ASSIGNMENT 2
SOLUTIONS

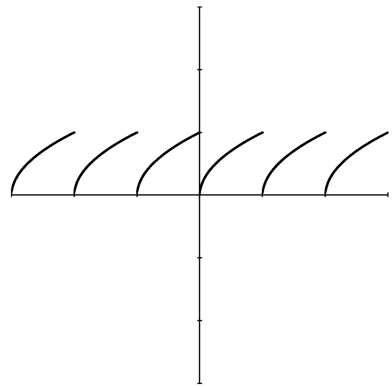
1. The symbol $[x]$ denote the largest integer which is $\leq x$. Thus, $[2.1] = [2] = 2$ and $[-0.9] = [-1] = -1$. Draw the graphs of the following functions.



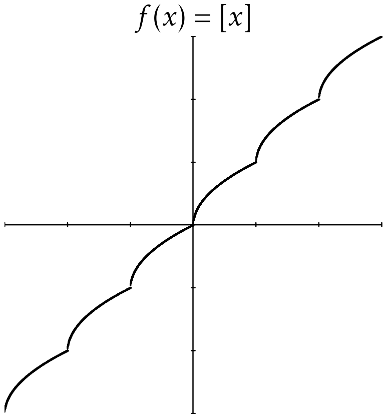
$f(x) = [x]$



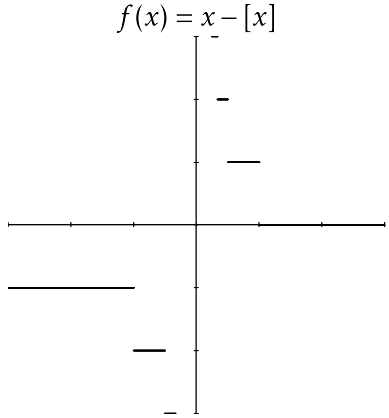
$f(x) = x - [x]$



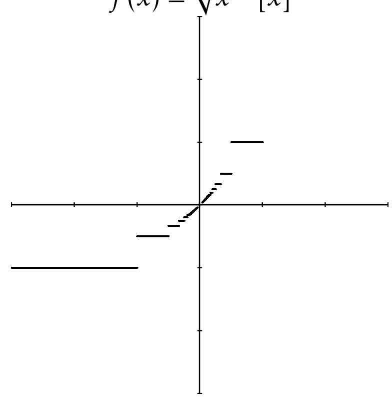
$f(x) = \sqrt{x - [x]}$



$f(x) = [x] + \sqrt{x - [x]}$



$f(x) = \left[\frac{1}{x}\right]$



$f(x) = \dfrac{1}{\left[\frac{1}{x}\right]}$

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	-4	4	2
4	-1	3	6	7

2. *The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table above gives the values of the functions and their first derivatives at selected values of x. The function h is given by $h(x) = f(g(x)) - 6$. Explain why there must be a value r for $1 < r < 3$ such that $h(r) = -5$.*

Since f and g are differentiable, they are continuous and so is h. Since $f(g(1)) = 9$ and $f(g(3)) = -1$. Thus, $h(1) = 3$ and $h(3) = -7$. Since $-7 < -5 < 3$, by the Intermediate Value Theorem, there must be some number between 1 and 3 so that $h(r) = -5$.

3. *Let f be a function defined by*

$$f(x) = \begin{cases} 2x - x^2 & \text{for } x \leq 1 \\ x^2 + kx + p & \text{for } x > 1 \end{cases}.$$

For what values of k and p will f be continuous and differentiable at $x = 1$?

For the function to be continuous, the limit at $x = 1$ must exist. This means that

$$1 = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = 1 + k + p.$$

For the function to be differentiable at $x = 1$, the derivative must exist there. So,

$$2 - 2x|_{x=1} = 0 = 2x + k|_{x=1} = k + 2.$$

Therefore, $k = -2$ and $p = 2$.

4. *Let f be the function defined as follows:*

$$f(x) = \begin{cases} |x+1| + 2 & \text{for } x < 1 \\ ax^2 + bx & \text{for } x \geq 1, \text{ where } a \text{ and } b \text{ are constants} \end{cases}$$

(a) *If $a = 24$ and $b = 3$, is f continuous for all x? Justify your answer.*

No, it is not continuous at $x = 1$ because $\lim_{x \to 1^-} f(x) = 4$ and $\lim_{x \to 1^+} f(x) = 27$, which means that the limit at $x = 1$ does not exist.

(b) *Describe all values of a and b for which f is a continuous function.*

For f to be continuous at $x = 1$, we must have that $\lim_{x \to 1^+} f(x) = a + b = 4 = \lim_{x \to 1^-} f(x)$. Thus, a and b lie on the line $x + y = 4$.

(c) *For what values of a and b is f both continuous and differentiable?*

This means that we must have a derivative at $x = 1$. For x close to 1, $|x+1| = x + 1$ so the derivative from the left is 1. The right hand derivative at $x = 1$ is $2a + b$. Therefore, we must have that

$$\begin{aligned} a + b &= 4 \\ 2a + b &= 1 \end{aligned}$$

Solving, gives us that $a = 7$ and $b = -3$.

5. *Let p and q be real numbers and let f be the function defined by:*

$$f(x) = \begin{cases} 1 + 2p(x-1) + (x-1)^2 & \text{for } x \leq 1 \\ qx + p & \text{for } x > 1 \end{cases}$$

(a) *Find the value of q, in terms of p for which f is continuous at x = 1.*

$$\lim_{x\to 1^-} f(x) = 1 = \lim_{x\to 1^+} f(x) = p + q.$$

Thus, we must have $q = 1 - p$.

(b) *Find the values of p and q for which f is differentiable at x = 1.*

$$f'_-(1) = 2p + 2(x-1)|_{x=1} = 2p.$$

$$f'_+(1) = q.$$

Thus, we have that $2p = q = 1 - p$ or $p = 1/3$ and $q = 2/3$.

(c) *If p and q have the values determined in part (b), is f'' a continuous function?*

$$f(x) = \begin{cases} 1 + \frac{2(x-1)}{3} + (x-1)^2 & \text{for } x \le 1 \\ \frac{2x+1}{3} & \text{for } x > 1 \end{cases}$$

$$f'(x) = \begin{cases} \frac{1}{3} + 2(x-1) & \text{for } x \le 1 \\ \frac{2}{3} & \text{for } x > 1 \end{cases}$$

$$f''(x) = \begin{cases} 2 & \text{for } x \le 1 \\ 0 & \text{for } x > 1 \end{cases}$$

Thus, $f''(x)$ is not continuous at $x = 1$.

6. *Let $f\ (a,b) \to \mathbb{R}$ be continuous, with $(a,b) \subseteq \mathbb{R}$. Show that if $f(r) = 0$ for each rational number $r \in (a,b)$, then $f(x) = 0$ for all $x \in (a,b)$.*

   Let $x \in \mathbb{R} \setminus \mathbb{Q}$. There is a sequence of rational numbers $\{x_n\}$ so that $\lim_{n\to\infty} x_n = x$. Since $f(x_n) = 0$ for all n and since f is continuous, $f(x) = \lim_{n\to\infty} f(x_n) = 0$ for all real numbers.

7. *Let $f(a,b) \to \mathbb{R}$ and $g(a,b) \to \mathbb{R}$ be continuous, with $(a,b) \subseteq \mathbb{R}$, so that $f(r) = g(r)$ for each rational number $r \in (a,b)$. Prove that $f(x) = g(x)$ for all $x \in (a,b)$.*

   Let $h(x) = f(x) - g(x)$. Then h is a continuous function which is 0 at all rational numbers. By the previous problem, $h(x) = 0$ for all $x \in \mathbb{R}$. Thus, $f(x) = g(x)$ for all real numbers.

8. *Let f and g be continuous functions on $[a,b]$ such that $f(a) \ge g(a)$ and $f(b) \le g(b)$. Prove that $f(x_0) = g(x_0)$ for some $x_0 \in [a,b]$.*

   Let $h(x) = f(x) - g(x)$. Then $h(a) = f(a) - g(a) \ge 0$ and $h(b0 = f(b) - g(b) \le 0$. Should either one of these values be zero, we have our point in question. Therefore, assume that $h(a) > 0$ and $h(b) < 0$. By the Intermediate Value Theorem, there is a point $x_0 \in (a,b)$ so that $h(x_0) = 0$. This means that $f(x_0) = g(x_0)$.

9. *Prove that $x2^x = 1$ for some $x \in (0,1)$.*

   Let $f(x) = x2^x$. f is a continuous function. $f(0) = 0$ and $f(1) = 2$. Therefore, but the Intermediate Value Theorem, there is a point $x \in (0,1)$ so that $f(x) = 1$.