MATH 6102 — SPRING 2009
ASSIGNMENT 2

February 2, 2009
Due February 9, 2009

1. The symbol $[x]$ denote the largest integer which is $\leq x$. Thus, $[2.1] = [2] = 2$ and $[-0.9] = [-1] = -1$. Draw the graphs of the following functions.

   (a) $f(x) = [x]$.

   (b) $f(x) = x - [x]$.

   (c) $f(x) = \sqrt{x - [x]}$.

   (d) $f(x) = [x] + \sqrt{x - [x]}$.

   (e) $f(x) = \left[\dfrac{1}{x}\right]$.

   (f) $f(x) = \dfrac{1}{\left[\dfrac{1}{x}\right]}$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	-4	4	2
4	-1	3	6	7

2. The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table above gives the values of the functions and their first derivatives at selected values of x. The function h is given by $h(x) = f(g(x)) - 6$. Explain why there must be a value r for $1 < r < 3$ such that $h(r) = -5$.

3. Let f be a function defined by

$$f(x) = \begin{cases} 2x - x^2 & \text{for } x \leq 1 \\ x^2 + kx + p & \text{for } x > 1 \end{cases}.$$

   For what values of k and p will f be continuous and differentiable at $x = 1$?

4. Let f be the function defined as follows:

$$f(x) = \begin{cases} |x+1| + 2 & \text{for } x < 1 \\ ax^2 + bx & \text{for } x \geq 1, \text{ where } a \text{ and } b \text{ are constants} \end{cases}$$

   (a) If $a = 24$ and $b = 3$, is f continuous for all x? Justify your answer.

   (b) Describe all values of a and b for which f is a continuous function.

(c) For what values of a and b is f both continuous and differentiable?

5. Let p and q be real numbers and let f be the function defined by:

$$f(x) = \begin{cases} 1 + 2p(x-1) + (x-1)^2 & \text{for } x \leq 1 \\ qx + p & \text{for } x > 1 \end{cases}$$

   (a) Find the value of q, in terms of p for which f is continuous at $x = 1$.
   (b) Find the values of p and q for which f is differentiable at $x = 1$.
   (c) If p and q have the values determined in part (b), is f'' a continuous function?

6. Let $f\ (a,b) \to \mathbb{R}$ be continuous, with $(a,b) \subseteq \mathbb{R}$. Show that if $f(r) = 0$ for each rational number $r \in (a,b)$, then $f(x) = 0$ for all $x \in (a,b)$.

7. Let $f\ (a,b) \to \mathbb{R}$ and $g\ (a,b) \to \mathbb{R}$ be continuous, with $(a,b) \subseteq \mathbb{R}$, so that $f(r) = g(r)$ for each rational number $r \in (a,b)$. Prove that $f(x) = g(x)$ for all $x \in (a,b)$.

8. Let f and g be continuous functions on $[a,b]$ such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Prove that $f(x_0) = g(x_0)$ for some $x_0 \in [a,b]$.

9. Prove that $x2^x = 1$ for some $x \in (0,1)$.