

ASSIGNMENT 3 SOLUTIONS

1. Express all the hyperbolic functions in terms of $\sinh x$. Given $\cosh x = 2$ find the values of the other functions.

$$\begin{aligned}\sinh x &= \sinh x \\ \cosh x &= \sqrt{\sinh^2 x + 1} \\ \tanh x &= \frac{\sinh x}{\sqrt{\sinh^2 x + 1}} \\ \coth x &= \frac{\sqrt{\sinh^2 x + 1}}{\sinh x} \\ \operatorname{sech} x &= \frac{1}{\sqrt{\sinh^2 x + 1}} \\ \operatorname{csch} x &= \frac{1}{\sinh x}\end{aligned}$$

Now, if $\cosh x = 2$, then $\sinh x = \sqrt{3}$ and

$$\begin{aligned}\sinh x &= \sqrt{3} \\ \cosh x &= 2 \\ \tanh x &= \frac{\sqrt{3}}{2} \\ \coth x &= \frac{2}{\sqrt{3}} \\ \operatorname{sech} x &= \frac{1}{2} \\ \operatorname{csch} x &= \frac{1}{\sqrt{3}}\end{aligned}$$

2. (a) Show that

$$(\cosh u_1 + \sinh u_1)(\cosh u_2 + \sinh u_2) = \cosh(u_1 + u_2) + \sinh(u_1 + u_2).$$

$$\begin{aligned}(\cosh u_1 + \sinh u_1)(\cosh u_2 + \sinh u_2) &= \cosh u_1 \cosh u_2 + \cosh u_1 \sinh u_2 + \\ &\quad \sinh u_1 \cosh u_2 + \sinh u_1 \sinh u_2 \\ &= (\cosh u_1 \cosh u_2 + \sinh u_1 \sinh u_2) + \\ &\quad (\sinh u_1 \cosh u_2 + \cosh u_1 \sinh u_2) \\ &= \cosh(u_1 + u_2) + \sinh(u_1 + u_2)\end{aligned}$$

(b) Show that for any positive integer $n > 0$

$$\prod_{i=1}^n (\cosh u_i + \sinh u_i) = \cosh \left(\sum_{i=1}^n u_i \right) + \sinh \left(\sum_{i=1}^n u_i \right).$$

This is done by induction with (a) serving as the first step. Assume that this is true for n . We need to prove it true for $n + 1$.

$$\begin{aligned} \prod_{i=1}^{n+1} (\cosh u_i + \sinh u_i) &= \left(\prod_{i=1}^n (\cosh u_i + \sinh u_i) \right) (\cosh u_{n+1} + \sinh u_{n+1}) \\ &= \left(\cosh \left(\sum_{i=1}^n u_i \right) + \sinh \left(\sum_{i=1}^n u_i \right) \right) (\cosh u_{n+1} + \sinh u_{n+1}) \end{aligned}$$

which by the above process

$$\begin{aligned} &= \cosh \left(\sum_{i=1}^n u_i + u_{n+1} \right) + \sinh \left(\sum_{i=1}^n u_i + u_{n+1} \right) \\ &= \cosh \left(\sum_{i=1}^{n+1} u_i \right) + \sinh \left(\sum_{i=1}^{n+1} u_i \right) \end{aligned}$$

which is what we needed to show.

(c) What does this become if $u_1 = u_2 = \cdots = u_n = u$?

This becomes an analogue of Euler's Formula:

$$(\cosh u + \sinh u)^n = \cosh(nu) + \sinh(nu).$$

3. Evaluate the following integral in terms of hyperbolic trigonometric functions

$$\int \frac{1}{\sqrt{4+x^2}} dx$$

Use the substitution $x = 2 \sinh u$, then $dx = 2 \cosh u du$ and the integral becomes:

$$\begin{aligned} \int \frac{1}{\sqrt{4+x^2}} dx &= \int \frac{1}{\sqrt{4 \cosh^2 u}} 2 \cosh u du \\ &= \int 1 du = u = \sinh^{-1} \left(\frac{x}{2} \right) + C. \end{aligned}$$

4. Differentiate the following functions.

(a) $f(x) = 3x \tanh(4x)$.

$$f'(x) = 3 \tanh(4x) + 12x \operatorname{sech}^2(4x).$$

(b) $g(x) = 5x \operatorname{sech}(4x) - 21 \tanh^3(4x).$

$$\begin{aligned} g'(x) &= 5 \operatorname{sech}(4x) - 20x \operatorname{sech}(4x) \tanh(4x) - 63 \tanh^2(4x) \cdot \operatorname{sech}(4x) \cdot 4 \\ &= 5 \operatorname{sech}(4x) - 20x \operatorname{sech}(4x) \tanh(4x) - 252 \tanh^2(4x) \operatorname{sech}(4x) \end{aligned}$$

5. (a) Use the substitution $x = \cosh u$, $u > 0$ to show that

$$\int \frac{1}{\sqrt{x^2 - 1}} dx = \cosh^{-1}(x) + C$$

for $x > 1$.

As in the previous integration problem, if $x = \cosh u$, then $dx = \sinh u du$ and

$$\begin{aligned} \int \frac{1}{\sqrt{x^2 - 1}} dx &= \int \frac{1}{\sqrt{\sinh^2 u}} \cosh u du \\ &= \int 1 du = u + C = \cosh^{-1} x + C \end{aligned}$$

- (b) Use the substitution $x = \sec u$, $0 < u < \frac{\pi}{2}$, to show that

$$\int \frac{1}{\sqrt{x^2 - 1}} dx = \ln \left| x + \sqrt{x^2 - 1} \right| + C$$

for $x > 1$.

Let $x = \sec u$, then $dx = \sec u \tan u du$ and

$$\begin{aligned} \int \frac{1}{\sqrt{x^2 - 1}} dx &= \int \frac{1}{\sqrt{\tan^2 u}} \sec u \tan u du \\ &= \int \sec u du = \ln |\sec u + \tan u| + C = \ln \left| x + \sqrt{1 + x^2} \right| + C \end{aligned}$$

- (c) Use the above to show that

$$\cosh^{-1}(x) = \ln \left| x + \sqrt{x^2 - 1} \right|$$

for $x > 1$.

Since both sides are antiderivatives for the same function, they differ by a constant, i.e.,

$$\cosh^{-1}(x) = \ln \left| x + \sqrt{x^2 - 1} \right| + k$$

We know that $\cosh^{-1}(1) = 0$ and $\ln \left| 1 + \sqrt{1 - 1} \right| = 0$, so we have $0 = 0 + k$, or $k = 0$ so that $\cosh^{-1}(x) = \ln \left| x + \sqrt{x^2 - 1} \right|$.

6. We define the function $\text{glog}(x) = y$ if and only if $x = \frac{e^y}{y}$. Find the derivative of $\text{glog}(x)$.

From the definition of $\text{glog}(x)$, we know that $x \cdot \text{glog}(x) = e^{\text{glog}(x)}$. Differentiating both sides with respect to x gives us

$$\begin{aligned} \text{glog}(x) + x \frac{d}{dx} \text{glog}(x) &= \frac{d}{dx} \text{glog}(x) e^{\text{glog}(x)} \\ \frac{d}{dx} \text{glog}(x) (e^{\text{glog}(x)} - x) &= -\text{glog}(x) \\ \frac{d}{dx} \text{glog}(x) &= -\frac{\text{glog}(x)}{(e^{\text{glog}(x)} - x)} \\ \frac{d}{dx} \text{glog}(x) &= -\frac{\text{glog}(x)}{x(\text{glog}(x) - 1)} \end{aligned}$$

7. Find the second derivative of the Lambert W function.

We showed that

$$W'(x) = \frac{W(x)}{x(1+W(x))}$$

so we differentiate this to find

$$\begin{aligned} W''(x) &= \frac{x(1+W(x))W'(x) - W(x)(1+W(x) + xW'(x))}{(x(1+W(x)))^2} \\ &= \frac{xW'(x) - W(x) - W^2(x)}{(x(1+W(x)))^2} \\ &= -\frac{W^2(x)(2+W(x))}{(1+W(x))^3 x^2} \\ &= \frac{W(x)}{(1+W(x))^2 x^2} - \frac{W^2(x)}{x^2(1+W(x))^3} - \frac{W(x)}{x^2(1+W(x))} \end{aligned}$$

8. Using the Lambert W function, solve $x \log_b x = a$ for x .

Rewrite this equation as

$$x \log_b x = a \Leftrightarrow \log_b x = a/x \Leftrightarrow x = b^{a/x} \Leftrightarrow x b^{-a/x} = 1.$$

Use the defining equation for the Lambert W function:

$$x = W(y) \implies y = x e^x.$$

So we want to get $x b^{-a/x}$ into the form $u e^u$. We know that $b^x = e^{x \ln b}$ and $b^{-a/x} = e^{-\frac{x \ln b}{a}}$, so we can multiply both sides by $-\frac{\ln b}{a}$ to get

$$\begin{aligned} x b^{-a/x} &= 1 \\ -\frac{x \ln b}{a} \cdot e^{-x \frac{\ln b}{a}} &= -\frac{\ln b}{a} \end{aligned}$$

is true if and only if

$$-\frac{x \ln b}{a} = W\left(-\frac{\ln b}{a}\right)$$
$$x = -\frac{aW\left(-\frac{\ln b}{a}\right)}{\ln b}$$