ASSIGNMENT 4
SOLUTIONS

1. *The number a is called a **double root** of the polynomial function f if $f(x) = (x-a)^2 g(x)$ for some polynomial function g. Prove that a is a double root of f if and only if a is a root of f and f'.*

   Assume that a is a double root of f. That means that $f(x) = (x-a)^2 g(x)$ and $f'(x) = 2(x-a)g(x) + (x-a)^2 g'(x) = (x-a)[2g(x) + (x-a)g'(x)]$ and a is a root of f'.

   Assume that a is a root of f and f'. That means that $f(x) = (x-a)g(x)$ and $f'(x) = (x-a)h(x)$ where g and h are polynomial functions.

$$\begin{aligned} f'(x) &= g(x) + (x-a)g'(x) \\ (x-a)h(x) &= g(x) + (x-a)g'(x) \\ g(x) &= (x-a)\left(h(x) - g'(x)\right) \end{aligned}$$

   Since h and g are polynomial functions, then so is $h - g'$ and we have that $g(x) = (x-a)p(x)$, which implies that $f(x) = (x-a)^2 p(x)$ and a is a double root of f.

2. *Prove that it is impossible to write $x = f(x)g(x)$ where f and g are differentiable and $f(0) = g(0) = 0$.*

   Assume that we can, *i.e.*, there exist differentiable functions f and g so that $x = f(x)g(x)$ and $f(0) = 0 = g(0)$. Then

$$\begin{aligned} x &= f(x)g(x) \\ 1 &= f(x)g'(x) + f'(x)g(x) \text{ for all } x \end{aligned}$$

   Thus, for $x = 0$

$$\begin{aligned} 1 &= f(0)g'(0) + f'(0)g(0) \\ 1 &= 0 \end{aligned}$$

   This contradiction indicates that no two such functions can exist.

3. *Suppose that f is differentiable at a. Prove that*

$$f'(a) = \lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h}.$$

   If f is differentiable at a, then

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

   Replacing a by $a-h$ does not change the limit, so

$$f'(a) = \lim_{h\to 0} \frac{f(a) - f(a-h)}{h}.$$

Therefore,

$$\begin{aligned} 2f'(a) = f'(a) + f'(a) &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} + \lim_{h\to 0} \frac{f(a)-f(a-h)}{h} \\ &= \lim_{h\to 0}\left[\frac{f(a+h)-f(a)}{h} + \frac{f(a)-f(a-h)}{h}\right] \\ 2f'(a) &= \lim_{h\to 0}\left[\frac{f(a+h)-f(a-h)}{h}\right] \\ f'(a) &= \lim_{h\to 0}\left[\frac{f(a+h)-f(a-h)}{2h}\right] \end{aligned}$$

4. *Find the first derivative of each of the following:*

   (a) $\sin(\cos(x))$

   $$\frac{d}{dx}\sin(\cos x) = -\cos(\cos x)\sin x$$

   (b) $\sin\left(\frac{\cos x}{x}\right)$

   $$\frac{d}{dx}\sin\left(\frac{\cos x}{x}\right) = -\cos\left(\frac{\cos x}{x}\right)\left(\frac{\sin x}{x} + \frac{\cos x}{x^2}\right).$$

   (c) $\frac{\sin(\cos x)}{x}$

   $$\frac{d}{dx}\frac{\sin(\cos x)}{x} = -\frac{\cos(\cos x)\sin x}{x} - \frac{\sin(\cos x)}{x^2}$$

5. *Find $f(f'(x))$ if $f(x) = \frac{1}{x}$.*

   $$f'(x) = -\frac{1}{x^2}, \text{ so } f(f'(x)) = \frac{1}{-1/x^2} = -x^2.$$

6. *Let $f(x)$ be differentiable on $(0,\infty)$ and assume that $\lim_{x\to\infty}[f(x)+f'(x)] = L$.*

   (a) *Find $\lim_{x\to\infty} f(x)$.* Let $h(x) = \frac{e^x f(x)}{e^x}$. Then, $h(x) = f(x)$ for all $x \in (0,\infty)$ at which $f(x) \neq 0$. Using l'Hospital's Rule on h, we find

   $$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} h(x) = \lim_{x\to\infty} \frac{e^x f(x)}{e^x}$$

   By l'Hospital's Rule

   $$= \lim_{x\to\infty} \frac{e^x(f(x)+f'(x))}{e^x} = \lim_{x\to\infty}[f(x)+f'(x)] = l$$

   (b) *If L is a number, find $\lim_{x\to\infty} f'(x)$.* Note that $f'(x) = [f(x)+f'(x)] - f(x)$. Thus, if L is a number, then

   $$\lim_{x\to\infty} f'(x) = \lim_{x\to\infty}([f(x)+f'(x)] - f(x)) = \lim_{x\to\infty}[f(x)+f'(x)] - \lim_{x\to\infty} f(x) = L - L = 0$$

   This last equality requires that L be a number.

7. *If f is differentiable on $(0,\infty)$ and $\lim_{x\to\infty} f'(x) = L$, and $\lim_{n\to\infty} f(n)$ exists as a number, then what must be the value of L?*

Consider $\dfrac{f(x)}{x}$.

$$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} f'(x)1 = L.$$

Thus,

$$L = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{n\to\infty} \frac{f(n)}{n}.$$

Now, we are given that $\lim_{n\to\infty} f(n)$ is a number, therefore

$$\lim_{n\to\infty} \frac{f(n)}{n} = 0 = L.$$