The Derivative and Differentiation

This is meant to be a theoretical treatment of differentiation and all of its related concepts. These would have been covered in a standard Calculus course, but here we will endeavor to include proofs of the main results.

I. Rates of Change and Tangent Lines

When a particle or a person is moving, or in motion, we can measure the ratio of how far he/she/it has traveled to how long it takes to travel that distance. This ratio is called the average speed. The units of measure are units of length divided by units of time – miles per hour, feet per second, furlongs per fortnight, or whatever is important to the problem at hand.

Below are some examples of different average speeds.
Notation: m/s = meters per second; km/h = kilometers per hour; mph = miles per hour; fps = feet per second

1. Speed of a common snail: 0.001 m/s = 0.0036 km/h = 0.0023 mph = 0.0188 inches per second.
2. A brisk walk = 1.667 m/s = 6 km/h = 3.75 mph.
3. Olympic sprinters (average speed over 100 meters) = 10.23 m/s = 36.85 km/h = 23.11 mph.
4. Speed limit on an interstate highway = 28.794 m/s = 103.66 km/h = 65 mph = 44.32 fps.
5. Top cruising speed of a Boeing 747-8 = 290.947 m/s = 1047.41 km/h = 650.83 mph : (officially Mach 0.85)
6. Official air speed record = 980.278 m/s = 3,529 km/h = 2,188 mph.
7. Space shuttle on re-entry = 7,777.778 m/s = 28,000 km/h = 17,500 mph.
8. the speed of sound in dry air at 70°F (Mach 1): 344 m/s = 1238 km/h = 769 mph = 1128 fps
9. The speed of sound in water at 25°C: 1,497 m/s = 5,389.2 km/h = 3,379.31 mph
10. The elevators in the Sears Tower, Chicago (1451 ft): 1600 ft/min
11. Highest surface wind speed recorded on earth: 231 mph at Mt. Washington Observatory, NH
12. F-22A Raptor supersonic fighter: Mach 2.5+, in excess of 1600 mph
13. Dodge Viper GTS Coupe: 177 mph
14. 81 mph – fastest human powered bicycle speed on flat ground
15. Top speed of the cheetah – 70 mph
16. Top speed of the Thompson's gazelle (the cheetah's favorite food) – 50 mph
17. Top speed of the Peregrine falcon – 200 mph
18. Top speed of the Sailfish (genus *Istiphorus*) – 68.5 mph

There are clearly many different average speeds to consider. Let's say that you are climbing at Hanging Rock State Park near Danbury, NC. A rock breaks loose from one of the cliffs. What would be the average speed during the first 2 seconds of the fall?

From physics we know that a dense solid object dropped from rest that falls freely near the surface of the earth will fall according to the equation:

$$y = 16t^2 \text{ feet}$$

in the first *t* seconds. Thus the average speed of the rock over the first two seconds will be the distance it falls, Δy, divided by the time differential, Δt, so for the average speed we get

$$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(0)^2}{2-0} = 32 \text{ ft/sec}$$

This tells us the average speed. However, we know that the rock is picking up speed as it falls, so this does not tell us what the speed is at any one moment in time. For that we want to find the instantaneous speed. That is, we need to find the speed over a smaller and smaller time period. So, at the instant when *t*=2, the speed of the rock will be about equal to the average speed over a very short time period, or between *t*=2 and $t = 2 + h$ for a very small number *h*.

$$\frac{\Delta y}{\Delta t} = \frac{16(2+h)^2 - 16(2)^2}{(2+h)-2} = \frac{16(2+h)^2 - 16(2)^2}{h} \text{ ft/sec}$$

We cannot use this to find the speed at *t*=2 because that would mean that we would have to take *h*=0, and this would give us 0/0, which is undefined. Thus, we want to know the value of this quotient when *h* is close to 0 and getting closer to 0, or

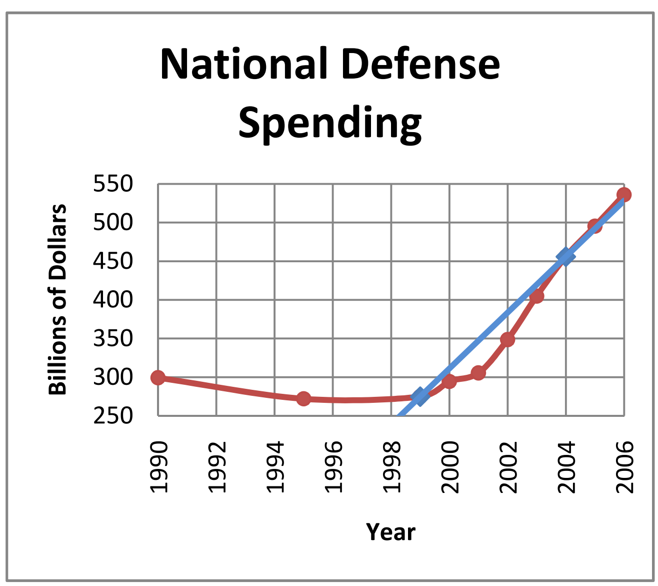
$$\lim_{h \to 0} \frac{16(2+h)^2 - 16(2)^2}{(2+h)-2} = \lim_{h \to 0} \frac{16(2+h)^2 - 16(2)^2}{h} = \lim_{h \to 0} (64 + 16h) \text{ ft/sec}$$

I.1: Average Rate of Change

For a given function $f(x)$ we can compute the average rate of change of the function over the interval [*a*,*b*] by the difference quotient:

$$\frac{f(b) - f(a)}{b - a}$$

Year	National Defense Spending ($ billion)
1990	299.3
1995	272.1
1999	274.9
2000	294.5
2001	305.5
2002	348.6
2003	404.9
2004	455.9
2005	495.3
2006	535.9



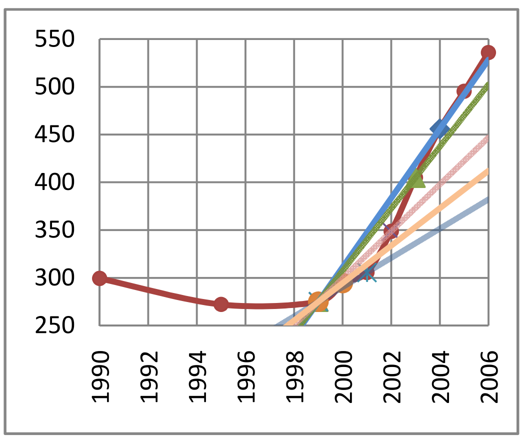
In the table and graph above we see the National Defense Spending in a number of different years since 1990. We can find the average rate of change in the defense spending between 1999 and 2004. If we let $P = (1999, 274.39)$ and $Q = (2004, 455.9)$, then this rate of change is given by the quotient

$$\textit{Average rate of change}: \frac{\Delta y}{\Delta x} = \frac{455.9 - 274.9}{2004 - 1999} = \frac{181}{5} = 36.2 \text{ billion dollars per year}$$

Note that this is the slope of the secant line joining *P* to *Q*. In fact, *we can always think of the average rate of change as being the slope of a secant line.*

We might want to know the average rates of change between 1999 and 2003, or between 1999 and 2002, etc. These are the slopes of the appropriate secant lines. We can compute these slopes, or average rates of change.

Q	Slope of $PQ = \Delta y / \Delta x$
(2003,404.9)	(404.9 – 274.9) / (2003 – 1999) = 32.5
(2002,348.6)	(348.6 – 274.9) / (2002 – 1999) = 24.567
(2001,305.5)	(305.5 – 274.9) / (2001 – 1999) = 15.3
(2000,294.5)	(294.5 – 274.9) / (2000 – 1999) = 19.6



As we let *Q* get closer and closer to the *P*, we see that the secant lines get closer and closer to the line that is tangent to the curve at the point *P*. If we were to sketch in a line that we think might approximate the tangent line, we might find that it passes through (1994,250) and (2006,310) and the slope of that line is 5 billion dollars per year. This, of course is strictly an approximation. It does, however, give us the idea behind what we might want to define for the tangent line and how we might be able to compute its slope. We clearly have the point $(a, f(a))$ through which the tangent line would pass. The only other information we need in order to determine uniquely that tangent line would be its slope. If the slope were m_T, then the equation of the tangent line would be $y - f(a) = m_T(x - a)$ or $y = f(a) + m_T(x - a)$.

I.2: Tangent Line to a Curve

We use the terminology from geometry. Given a line and a circle, there are exactly three ways they might intersect:

1. They can intersect in no points
2. They can intersect in one point, in which case we say that the line is *tangent* to the circle.
3. They can intersect in two points, in which case we call the line a *secant line*.

Now, for the most part, calculus is the study of local phenomena. What we mean by that is that we study what happens over small intervals around certain points. We do, occasionally, study *global* phenomena, but usually we are interested in what is going on in a small neighborhood. In our definition of secants and tangents we really need to accede to that point of view. We are going to choose a point $(a, f(a))$ on the curve. In terms of nearby points, or in a small neighborhood, a *secant line* to the curve is a line that passes through $(a, f(a))$ and $(b, f(b))$, where b is usually relatively close to a. A tangent line to $f(x)$ at $x=a$ is the limit of the secant lines as b approaches a. Visually this is a line the touches the graph of $y = f(x)$ only once in a small neighborhood about a, but does so in a manner that tends to behave like $f(x)$.

How can we see this? How can we make this more visual? Think about the graph of your favorite function. Let's say that we want to look at the graph of $f(x)$ on [-10,10]:

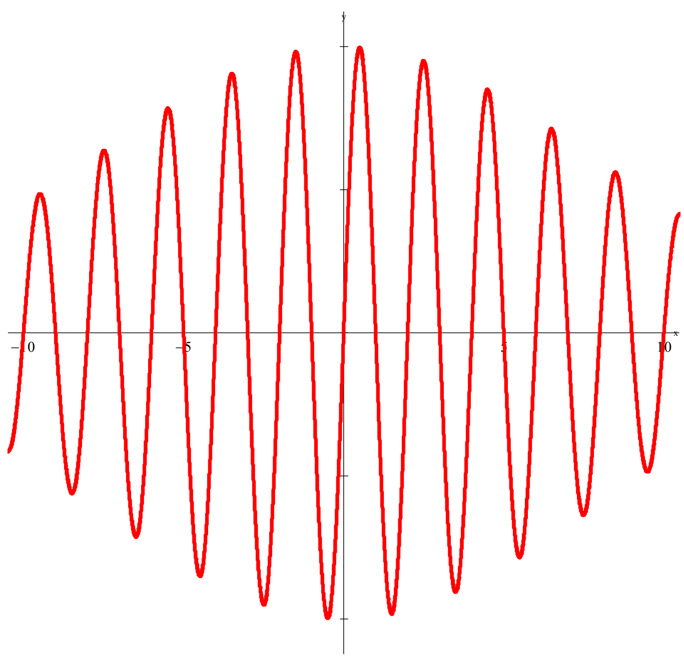
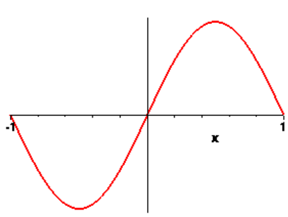
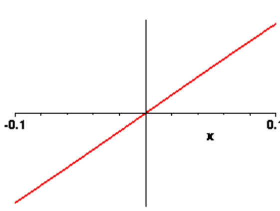


Figure 1: $f(x) = 2\sin(\pi x)e^{-x^2/125}$

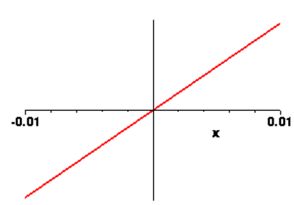
Now, this graph looks pretty "wiggly". It oscillates quickly and with different heights. We are interested in what is happening around x=0. So let's zoom in on the graph a few times – each time by a factor of 10. Thus, our first zoom will be to the interval [-1,1] and then to [-0.1,0.1], and so on.



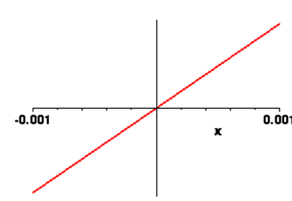
$f(x)$ on [-1,1]



$f(x)$ on [-0.1,0.1]



$f(x)$ on [-0.01,0.01]

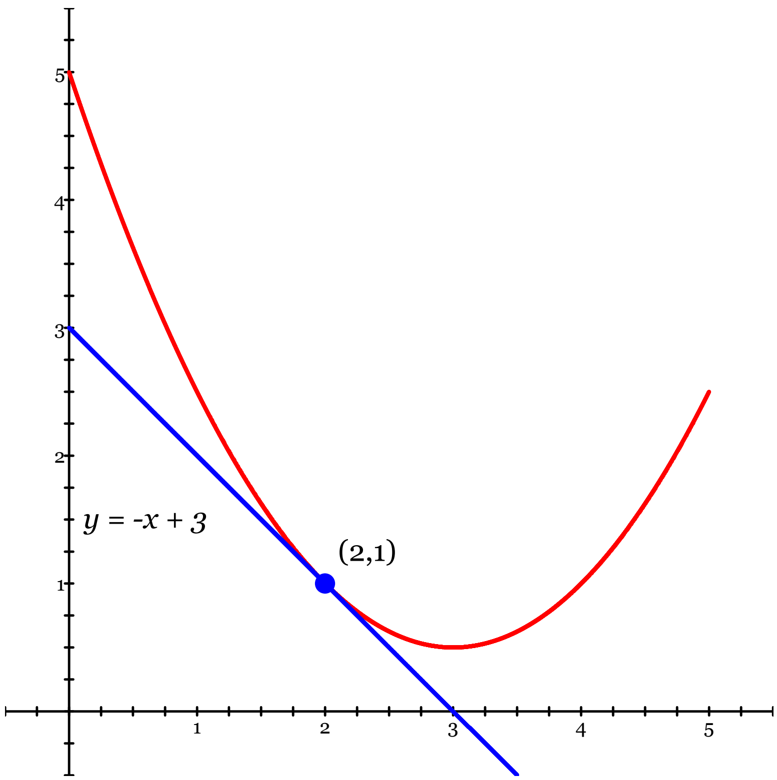


$f(x)$ on [-0.001,0.001]

As we zoom in we see what happens to the graph – it "flattens out". It begins to look like a line. By the time we are down to the interval [−0.1,0.1] it already looks like a line, and this is certainly true by the time we are looking on the interval [−0.001,0.001]! The graph looks *linear*, and we describe this by saying that the function is *locally linear* if this happens. This line that the function looks like should be the tangent line, if there is one.

Let's see if we can't compute one of these. Let's look at the function $f(x) = \frac{1}{2}x^2 - 3x + 5$ at the point $(2, f(2)) = (2,1)$. The graph of this function on the interval [0,5] looks like the following.

Let's compute the slope of the secant line that runs from (2,1) to the point $(2+h, f(2+h))$.



$$\text{Secant slope} = \frac{f(2+h) - f(2)}{h}$$

$$= \frac{(\frac{1}{2}(2+h)^2 - 3(2+h) + 5) - 1}{h}$$

$$= \frac{\frac{1}{2}h^2 - h}{h} = \frac{1}{2}h - 1$$

The limit of the secant slope as Q approaches P along the curve is the slope of the tangent line. In this case we get

$$\lim_{Q \to P}(\text{secant slope}) = \lim_{h \to 0}\left(\frac{1}{2}h - 1\right) = -1$$

.

Thus the slope of the parabola at (2,1) is −1. The tangent line to the curve at P is the line through (2,1) with slope *m*=−1.

$$y - 1 = -1(x - 2)$$
$$y = -x + 3$$

The *slope of a curve* $y = f(x)$ at the point $(a, f(a))$ is the number

$$\lim_{h \to 0}\frac{f(a+h) - f(a)}{h} = \lim_{x \to a}\frac{f(x) - f(a)}{x - a}$$

if the limit exists. The *tangent line to the curve* at the point is the line through the point $(a, f(a))$ with the above slope.

II Definition of the Derivative

In the last section we defined the slope of a curve $y = f(x)$ at the point $x = a$ to be the limit of the difference quotient:

$$m = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

When this limit exists, that number is called the derivative of *f* at *a*. We can study this phenomenon at each point in the domain of *f* at which the limit exists.

Definition: *The derivative of the function f is the function f' whose value at each point x in the domain of f is*

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

provided that the limit exists.

Note that the domain of f' is the set of points in the domain of *f* at which the limit exists. This may be strictly smaller than the domain of *f*, but can be no larger than the domain of *f*. If $f'(x)$ exists, then we say that *f* is *differentiable* at *x*. A function that is differentiable at each point in its domain is a differentiable function.

We denote the derivative of *f* in a couple of different ways. Each notation has its strong points and its drawbacks. The most common notation for the derivative of *f* at *x* is $f'(x)$, and is often called "*f* prime of *x*". This is very close to the notation used by Newton when he worked with "the calculus". His notation is still used by some physicists today. His notation for the derivative of *f* at *x* is $\dot{f}(x)$ — that is a small dot over *f*. It can be hard to see, so most books will use the "prime notation". The notation that Leibniz introduced is used today as well. His notation for the derivative of *f* at *x* is $\frac{df}{dx}(x) = \frac{d}{dx}f(x)$. This is sometimes read as "*df* by *dx*". It is not a fraction. A more modern notation introduced in the twentieth century is the notation *Df* or $D_x f$. Thus, we have

$$f'(x) = \dot{f}(x) = \frac{df}{dx}(x) = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

Example: Find the derivative of $f(x) = x^2 - 4x - 4$.

To do this we need to evaluate the limit of the difference quotient.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2 - 4(x+h) - 4 - (x^2 - 4x - 4)}{h}$$

$$= \lim_{h\to 0}\frac{2xh + h^2 - 4h}{h} = \lim_{h\to 0}(2x + h - 4) = 2x - 4$$

So we find that $f'(x) = 2x - 4$

Alternative Definition: *If we use the alternative definition of the slope of the tangent being the limit of the slopes of the secant lines, then we have that the derivative of f at the point $x = a$ is given by*

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

provided that the limit exists.

We find that this is usually the formulation that the derivative takes in most applications. We will be able to recognize certain problems as derivatives by seeing this type of limit in the process of setting up the problem.

<u>Example 1</u>: We can still evaluate the derivative using this alternative definition. Let's see that we would get the same thing as before, so let $f(x)=x^2-4x-4$.

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{x^2-4x-4-(a^2-4a-4)}{x-a}$$

$$=\lim_{x\to a}\frac{(x^2-a^2)-4(x-a)}{x-a}=\lim_{x\to a}\frac{(x-a)(x+a)-4(x-a)}{x-a}=\lim_{x\to a}\frac{(x-a)(x+a-4)}{x-a}$$

Now, as long as we get close to a, but never let x equal a, we have that the term $x–a$ in the numerator and denominator factor out. Thus,

$$f'(a)=\lim_{x\to a}\frac{(x-a)(x+a-4)}{x-a}=\lim_{x\to a}(x+a-4)=2a-4$$

which is what we got before.

Note that this process requires that we try find a factor of $x–a$ in the numerator to factor out with the denominator, whereas in the first definition we just had to find a common factor of h in each term of the numerator. Algebraically speaking, the first process may be easier, but the second has it uses.

Example 2: Find the derivative of $f(x)=\sqrt{x}$ using the alternative definition.

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{x-a}$$

$$=\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{x-a}=\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{x-a}\cdot\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}=\lim_{x\to a}\frac{x-a}{(x-a)(\sqrt{x}+\sqrt{a})}$$

$$f'(a)=\lim_{x\to a}\frac{x-a}{(x-a)(\sqrt{x}+\sqrt{a})}=\lim_{x\to a}\frac{1}{\sqrt{x}+\sqrt{a}}=\frac{1}{2\sqrt{a}}=\frac{1}{2}a^{-\frac{1}{2}}$$

Example 3: Find the derivative of $f(x)=x^3$ using the first definition.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^3-x^3}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h}=\lim_{h\to 0}(3x^2+3xh+h^2)=3x^2$$

III. Local Linearity

Since we have been using graphing calculators extensively in early courses are familiar with the principle of local linearity. We know from much experience that if we *zoom in* on just about any section of one of the functions we have studied, the function quickly *becomes linear*. our study of calculus adds to this base by explicitly expressing this *zoomed-in line* as the tangent to the curve at $x = a$, $y = f'(a)(x-a)+f(a)$.

Local linearity is a property of differentiable functions that says that if you zoom in on a point on the graph of the function (with equal scaling horizontally and vertically), the graph will eventually look like a straight line with a slope equal to the derivative of the function at the point.

Thus, local linearity is the graphical manifestation of differentiability.

Functions that are locally linear are smooth. Functions are not locally linear at points where they have discontinuities: breaks, jumps, vertical asymptotes, or the like. For example, $y = |x|$ is not locally linear at the origin.

Functions that are differentiable at a point are locally linear there, and *vice versa*. Unfortunately, there is no other definition of local linearity. It would be helpful if one could determine whether a function is locally linear at a point and then be able to conclude that is it differentiable. This is not possible. Since *zooming in* is accomplished using technology, such as a graphing calculator or computer graphing program, one can never be certain that one has zoomed in far enough. Thus if your function is given to you numerically, you can never be sure that it is differentiable.

On the other hand, if you have an analytic representation of your function (*i.e.* a formula containing standard functions like powers, trigonometric functions, *etc.*), then you can check to see if the function is differentiable at the point of interest. Namely, take the derivative at the point of interest and see if it is finite, and if so, it is differentiable and hence locally linear.

IV. Basic Properties of the Derivative

Definition Let f be a real-valued function defined on an open interval containing the point a. We say the f is *differentiable at a*, or that f *has a derivative at a*, if the limit

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

exists and is finite. We will write $f'(a)$ for the derivative of f at a:

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

whenever this limit exists and is finite.

We will want to speak about f' as function in its own right. The domain of f' is the set of points at which f has a derivative, so $\operatorname{dom}(f')\subseteq\operatorname{dom}(f)$.

The algebra of derivatives turns out to be pretty simple.

Example 1: The derivative of $f(x)=x^2-1$ at $x=3$ is given by

$$f'(3)=\lim_{x\to 3}\frac{f(x)-f(3)}{x-3}=\lim_{x\to 3}\frac{(x^2-1)-8}{x-3}=\lim_{x\to 3}x+3=6.$$

It is not much more difficult to compute the derivative at $x=a$:

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{(x^2-1)-(a^2-1)}{x-a}=\lim_{x\to a}x+a=2a.$$

This computation is valid for all real numbers a, so the function $f'(x)=2x$ is the derivative function of the function $f(x)=x^2-1$.

Example 2: Let $n\in\mathbb{Z}^+$, and let $f(x)=x^n$ for all $x\in\mathbb{R}$. Let $a\in\mathbb{R}$, then $f(x)-f(a)=x^n-a^n=(x-a)(x^{n-1}+ax^{n-2}+a^2x^{n-3}+\ldots+a^{n-2}x+a^{n-1})$. Therefore, $\dfrac{f(x)-f(a)}{x-a}=x^{n-1}+ax^{n-2}+a^2x^{n-3}+\ldots+a^{n-2}x+a^{n-1}$, for $x\neq a$. Thus,

$$\begin{aligned}f'(a)&=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\\&=a^{n-1}+aa^{n-2}+a^2a^{n-3}+\ldots+a^{n-2}a+a^{n-1}=na^{n-1}.\end{aligned}$$

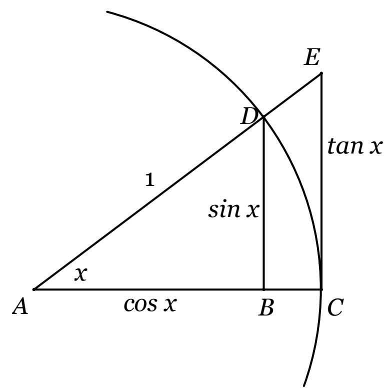


Figure 2

We want to find the derivative of the function $f(x)=\sin(x)$. This will require the following lemma.

Lemma 1: $\lim_{x\to 0}\dfrac{\sin x}{x}=1$.

Proof: In Figure 1 we are looking at a section of the unit circle. The area, A_I, of the inner triangle, ΔABD, has area $\frac{1}{2}\sin x\cos x$. The outer triangle, ΔACE, has area $A_O=\frac{1}{2}\tan x$.

The sector of the circle defined by A, C, and D has area $A_S=\dfrac{x}{2}$. We clearly see that $A_I<A_S<A_O$. This means that

$$\begin{aligned}\frac{\sin x\cos x}{2}&<\frac{x}{2}<\frac{\tan x}{2}\\ \sin x\cos x&<x<\tan x\\ \cos x&<\frac{x}{\sin x}<\frac{1}{\cos x}\\ \cos x&<\frac{\sin x}{x}<\frac{1}{\cos x}\end{aligned}$$

Thus,

$\lim_{x\to 0}\cos x \le \lim_{x\to 0}\frac{\sin x}{x} \le \lim_{x\to 0}\frac{1}{\cos x}$ which leads to $1 \le \lim_{x\to 0}\frac{\sin x}{x} \le 1$. Therefore, we have that $\lim_{x\to 0}\frac{\sin x}{x} = 1$.

Now, to find the derivative of the sine function we have:

$$
\begin{aligned}
\lim_{x\to a}\frac{\sin x - \sin a}{x-a} &= \lim_{x\to a}\frac{2\sin(\frac{x-a}{2})\cos(\frac{x+a}{2})}{x-a} \\
&= \lim_{x\to a}\frac{\sin(\frac{x-a}{2})\cos(\frac{x+a}{2})}{\frac{x-a}{2}} \\
&= \lim_{x\to a}\frac{\sin(\frac{x-a}{2})}{\frac{x-a}{2}}\cos(\frac{x+a}{2}) \\
&= \lim_{u\to 0}\frac{\sin(u)}{u}\lim_{u\to 0}\cos(u+a) \\
&= 1\cdot\cos a = \cos a
\end{aligned}
$$

Thus, we get that the derivative function for sin *x* is cos *x*.

We can show that the derivative function of the cosine function is –sin *x* in a similar manner.

$$
\begin{aligned}
\lim_{x\to a}\frac{\cos x - \cos a}{x-a} &= \lim_{x\to a}\frac{-2\sin\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right)}{x-a} \\
&= \lim_{x\to a}-\frac{\sin\left(\frac{x-a}{2}\right)}{\frac{x-a}{2}}\sin\left(\frac{x+a}{2}\right) \\
&= \lim_{x\to a}-\frac{\sin\left(\frac{x-a}{2}\right)}{\frac{x-a}{2}}\lim_{x\to a}\sin\left(\frac{x+a}{2}\right) \\
&= -1\cdot\sin a = -\sin a
\end{aligned}
$$

One of the first things we should notice is that differentiability at a point implies continuity at a point.

Theorem 1: *If f is differentiable at x = a, then f is continuous at x = a.*

Proof: Since *f* is differentiable at $x = a$, we know that

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

exists and is finite. We need to show that $\lim_{x\to a} f(x) = f(a)$. We have $f(x) = (x-a)\dfrac{f(x)-f(a)}{x-a} + f(a)$ for all $x \neq a$ in the domain of f. Taking the limit of both sides as $x \to a$ we get

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a}\left[(x-a)\frac{f(x)-f(a)}{x-a} + f(a)\right] \\ &= \lim_{x\to a}\left[(x-a)\frac{f(x)-f(a)}{x-a}\right] + \lim_{x\to a} f(a) \\ &= \lim_{x\to a}(x-a)\lim_{x\to a}\frac{f(x)-f(a)}{x-a} + f(a) \\ &= 0\cdot f'(a) + f(a) = f(a)\end{aligned}$$

We are done. ■

Note that the other direction is not true. What is a good counterexample?

V. Rules of Differentiation

We now want to look at the basic rules of differentiation. We will start with the easy Sum and Difference Rules.

Theorem 2: *If f and g are both differentiable, then f + g and f – g are differentiable and*

$$(f \pm g)'(x) = f'(x) \pm g'(x).$$

Proof: We will show that $(f+g)'(a) = f'(a) + g'(a)$ for an arbitrary $a \in \operatorname{dom} f$.

$$\begin{aligned}\lim_{x\to a}\frac{(f+g)(x)-(f+g)(a)}{x-a} &= \lim_{x\to a}\frac{f(x)+g(x)-(f(a)+g(a))}{x-a} \\ &= \lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a} + \frac{g(x)-g(a)}{x-a}\right] \\ &= f'(a) + g'(a)\end{aligned}$$

The difference is proven in exactly the same way. ■

This comes as no surprise.

Theorem 3: *If $c \in \mathbb{R}$ and f is the constant function given by f (x) = c for all $x \in \mathbb{R}$, then $f'(x) = 0$ for all $x \in \mathbb{R}$.*

Proof: Simply compute the derivative:

$$\begin{aligned}f'(a) &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \\ &= \lim_{x\to a}\frac{0}{x-a} = 0\end{aligned}$$

and we are done. ■

We need to compute three more general derivatives: the derivative of a product, the derivative of a quotient, and the derivative of a composition.

V.1 Product Rule

The *student product rule* would be what we would expect:

$$(fg)'(x) = f'(x)g'(x).$$

This is ***not*** true. Simple counterexamples are numerous.
The real problem lies in the difference quotient. ***Note*** that

$$\frac{f(x)g(x)-f(a)g(a)}{x-a} \neq \frac{f(x)-f(a)}{x-a}\cdot\frac{g(x)-g(a)}{x-a}.$$

Instead, note that

$$\begin{aligned}\frac{f(x)g(x)-f(a)g(a)}{x-a} &= \frac{f(x)g(x)-f(x)g(a)+f(x)g(a)-f(a)g(a)}{x-a} \\ &= f(x)\frac{g(x)-g(a)}{x-a}+g(a)\frac{f(x)-f(a)}{x-a}\end{aligned}$$

Now, taking the limit as $x \to a$ gives us that

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

This is the **Product Rule** for derivatives.

Note that the following is an immediate corollary of the previous two rules.

Corollary 1: $(cf)'(x) = c\cdot f'(x)$ if f is differentiable and $c \in \mathbb{R}$.

V.2 Quotient Rule

Theorem 4: *If f and g are differentiable at x = a and* $g'(a) \neq 0$, *then*

$$\left(\frac{f}{g}\right)'(a) = \frac{g(a)f'(a)-f(a)g'(a)}{(g(a))^2}.$$

Proof: Again, we just have to rewrite the quotient appropriately. Since $g'(a) \neq 0$ and g is continuous at $x = a$, there is an open interval I containing a so that $g(x) \neq 0$ for $x \in I$. Thus, for $x \in I$, we can write

$$\begin{aligned}(f/g)(x)-(f/g)(a) &= \frac{f(x)}{g(x)}-\frac{f(a)}{g(a)} \\ &= \frac{g(a)f(x)-f(a)g(x)}{g(x)g(a)} \\ &= \frac{g(a)f(x)-g(a)f(a)+g(a)f(a)-f(a)g(x)}{g(x)g(a)} \quad \text{so}\end{aligned}$$

$$\frac{(f/g)(x)-(f/g)(a)}{x-a} = \left[g(a)\frac{f(x)-f(a)}{x-a}-f(a)\frac{g(x)-g(a)}{x-a}\right]\frac{1}{g(x)g(a)}$$

Taking the limits as $x \to a$ gives us the result. ■

V.3 The Chain Rule

Theorem 5: *If f is differentiable at a and g is differentiable at f (a), then the composite function* $g \circ f$ *is differentiable at* a *and* $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$.

Proof: Since g is differentiable at $f(a)$ it can be shown that $g \circ f$ is defined on some open interval containing a. Let

$$h(y) = \frac{g(y) - g(f(a))}{y - f(a)}$$

for $y \in \operatorname{dom} g$ and $y \neq f(a)$, and let $h(f(a)) = g'(f(a))$. Since $\lim_{y \to f(a)} h(y) = h(f(a))$, the function h is continuous at $f(a)$.

$$g(y) - g(f(a)) = h(y)[y - f(a)]$$

for all $y \in \operatorname{dom} g$ so

$$(g \circ f)(x) - (g \circ f)(a) = h(f(x))[f(x) - f(a)] \text{ for } x \in \operatorname{dom}(g \circ f).$$

Therefore

$$\frac{(g \circ f)(x) - (g \circ f)(a)}{x - a} = h(f(x)) \frac{f(x) - f(a)}{x - a}$$

for $x \in \operatorname{dom}(g \circ f)$, $x \neq a$. Since $\lim_{x \to a} f(x) = f(a)$ and the function h is continuous at $f(a)$, $\lim_{x \to a} h(f(x)) = h(f(a)) = g'(f(a))$. The other limit in the above is $f'(a)$, so the result follows. ■

V.4 Other Transcendental Derivatives

We should find the derivatives of the natural logarithm and the exponential functions. However, for a strictly rigorous treatment we will defer until we do integration. For now, we will deal with these functions as follows.

First, recall that

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e.$$

Now, we want to compute the derivative of the natural logarithm, $\ln x$ which is defined by $\ln a = b$ means $a = e^b$. We know that this function satisfies all of the usual properties of the logarithms, so we begin with a lemma.

Lemma 2: *If f is differentiable at x = a, then*

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Then, if $f(x) = \ln x$ we have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h} \\
&= \lim_{h\to 0}\frac{1}{h}\ln\left(\frac{x+h}{x}\right) \\
&= \lim_{h\to 0}\frac{1}{h}\ln\left(1+\frac{h}{x}\right) \\
&= \lim_{h\to 0}\frac{1}{x}\frac{x}{h}\ln\left(1+\frac{h}{x}\right) \\
&= \lim_{h\to 0}\frac{1}{x}\ln\left[\left(1+\frac{h}{x}\right)^{x/h}\right] \\
&= \frac{1}{x}\lim_{t\to\infty}\ln\left(1+\frac{1}{t}\right)^{t} \\
&= \frac{1}{x}\ln\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{t} \\
&= \frac{1}{x}\ln(e) = \frac{1}{x}
\end{aligned}$$

Now, we made a number of assumptions here, such as the logarithm is a continuous function and others. Nonetheless, we are okay for now.

We have that the derivative of the natural logarithm of x is $1/x$. Now, to the exponential function. Here we will use the Chain Rule.

$$\begin{aligned}
\ln(e^x) &= x \\
\frac{d}{dx}\ln(e^x) &= \frac{d}{dx}x \\
\frac{1}{e^x}\frac{d}{dx}e^x &= 1 \\
\frac{d}{dx}e^x &= e^x
\end{aligned}$$

From these and the Chain Rule and the Product Rule the following are true:

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x \qquad \frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{d}{dx}\cot x = -\csc^2 x \qquad \frac{d}{dx}\sec x = \sec x\tan x \qquad \frac{d}{dx}\csc x = -\csc x\cot x$$

$$\frac{d}{dx}e^x = e^x \qquad \frac{d}{dx}a^x = a^x\ln a \qquad \frac{d}{dx}\ln x = \frac{1}{x}$$

$$\frac{d}{dx}\log_b x = \frac{1}{x\ln b} \qquad \frac{d}{dx}\sinh x = \cosh x \qquad \frac{d}{dx}\cosh x = \sinh x$$

Recall that we had defined the hyperbolic sine and cosine by

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}.$$

Recall that the Lambert W-function is defined by $W(a) = b$ means that $a = b\cdot e^b$. Thus, to find the derivative of this function we will use the Product Rule and the Chain Rule.

$$x = W(x)e^{W(x)}$$
$$\frac{dx}{dx} = \frac{d}{dx}\left(W(x)e^{W(x)}\right)$$
$$1 = W'(x)e^{W(x)} + W(x)e^{W(x)}W'(x)$$
$$W'(x) = \frac{1}{e^{W(x)}(1+W(x))}$$
$$\text{but } e^{W(x)} = x/W(x), \text{ so}$$
$$W'(x) = \frac{W(x)}{x(1+W(x))}, x \neq 0, -1/e$$

There are very few functions that we cannot differentiate using these theorems, if they have a derivative.

This technique leads us to a more general theorem.

Theorem 6: *Let f be a continuous one-to-one function defined on an interval and suppose that f is differentiable at* $a = f^{-1}(b)$*, with* $f'(a) \neq 0$*, then* f^{-1} *is differentiable at b and*

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}.$$

Proof: Let $b = f(a)$. Then

$$\lim_{h\to 0}\frac{f^{-1}(b+h) - f^{-1}(b)}{h} = \lim_{h\to 0}\frac{f^{-1}(b+h) - a}{h}.$$

Now, every number $b+h$ in the domain of f^{-1} can be written in the form $b + h = f(a + k)$ for a unique k. Then

$$\lim_{h\to 0}\frac{f^{-1}(b+h)-a}{h} = \lim_{h\to 0}\frac{f^{-1}(f(a+k))-a}{f(a+k)-b}$$
$$= \lim_{h\to 0}\frac{k}{f(a+k)-f(a)}$$

Now, since $b + h = f(a+k)$ we have

$$b+h = f(a+k)$$
$$f^{-1}(b+h) = a+k$$
$$k = f^{-1}(b+h) - a = f^{-1}(b+h) - f^{-1}(b)$$

The function f^{-1} is continuous at b, so we have that k approaches 0 as h approaches 0. Since

$$\lim_{k\to 0}\frac{f(a+k)-f(a)}{k}=f'(a)=f'(f^{-1}(b))\neq 0,$$

we have that

$$(f^{-1})'(b)=\frac{1}{f'(f^{-1}(b))}. \quad \blacksquare$$

This leads to the derivatives of the inverse trigonometric functions. Let's consider the arctangent function. First, we will choose the tangent with domain $(-\pi/2,\pi/2)$. The range is $\mathbb{R}$ and the arctangent will have domain $\mathbb{R}$ and range $(-\pi/2,\pi/2)$. I will use the notation **arctan** for the arctangent function.

$$\tan(\arctan x)=x$$

$$\sec^2(\arctan x)\frac{d}{dx}\arctan x=1$$

$$\frac{d}{dx}\arctan x=\cos^2(\arctan x)$$

However, because these are trigonometric functions, we know more. The arctangent of x is the angle whose tangent is x. The tangent of that angle can be found from a right triangle whose ratio of the opposite side to the adjacent side is x:1. Thus, we can take the opposite side to have length x and the adjacent side to have length 1. That means that the hypotenuse is $\sqrt{1+x^2}$. Then the cosine of that angle is the ratio of the adjacent side to the hypotenuse, or $1:\sqrt{1+x^2}$. Thus,

$$\cos(\arctan(x))=\frac{1}{\sqrt{1+x^2}}$$

and therefore

$$\frac{d}{dx}\arctan x=\frac{1}{1+x^2}.$$

Similar techniques show that

$$\frac{d}{dx}\arcsin x=\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\operatorname{arcsec} x=\frac{1}{x\sqrt{x^2-1}}$$

VI The Mean Value Theorem}

We know that a continuous function on a closed interval must take on its maximum and minimum values. How do we find these points? This is one of the places where the tool of calculus is helpful.

Theorem 7: *Let f is defined on an open interval containing c. If f assumes its maximum or minimum value at x = c, and if f is differentiable at x = c, then* $f'(c)=0$.

Proof: Suppose the *f* is defined on (a,b) with $a < c < b$. Since either *f* or $-f$ assumes its maximum at $x = c$, we may assume that *f* assumes its maximum at $x = c$. We need to show that $f'(c) = 0$.

Assume that $f'(c) > 0$. Since

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} > 0,$$

there is a $\delta > 0$ so that $a < c - \delta < c + \delta < b$ and if $0 < |x - c| < \delta$ then $\frac{f(x) - f(c)}{x - c} > 0.$

If we choose an $x \in (c, c + \delta)$, then from the above inequality we have that $f(x) > f(c)$, contrary to the assumption that *f* has its maximum at $x = c$. Likewise, if we assume that $f'(c) < 0$, there is a $\delta > 0$ so that if $0 < |x - c| < \delta$ then

$$\frac{f(x) - f(c)}{x - c} < 0.$$

If we choose an $x \in (c - \delta, c)$, then from the above inequality we have that $f(x) > f(c)$, contrary to the assumption that *f* has its maximum at $x = c$. Thus, we must have that $f'(c) = 0$. ■

<u>**Rolle's Theorem**</u>**:** *Let f be continuous on [a,b] that is differentiable on (a,b) and satisfies f (a) = f (b). Then there exists at least one* $c \in (a,b)$ *so that* $f'(c) = 0$.

Proof: We know that *f* takes its maximum and its minimum in the interval $[a,b]$. Thus, there exist points $x_0, y_0 \in [a,b]$ so that $f(x_0) \le f(x) \le f(y_0)$ for all $x \in [a,b]$. If x_0 and y_0 are both endpoints of $[a,b]$, then f must be a constant function since $f(a) = f(b)$ and $f'(x) = 0$ for all $x \in (a,b)$. Otherwise, f assumes either a maximum or a minimum at some point in the interior, (a,b), and by the above Theorem, $f'(x) = 0$. ■

This next theorem is a bit of a generalization of Rolle's Theorem. We can think of Rolle's Theorem as saying that somewhere in (a,b) there is a point at which the function has the same slope as the slope of the secant through $(a, f(a))$ and $(b, f(b))$, which is 0.

Mean Value Theorem: *Let f be a continuous function on [a,b] that is differentiable on (a,b). Then there is a point* $c \in (a,b)$ *such that* $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Proof: Let h be the function whose graph is the secant line through $(a, f(a))$ and $(b, f(b))$. Then we have that

$$h(a) = h(a), \quad h(b) = h(b), \quad h'(x) = \frac{f(b) - f(a)}{b - a}.$$

Let $g(x) = f(x) - L(x)$ for all $x \in [a,b]$. Then, g is continuous on $[a,b]$ and differentiable on (a,b). Note that $g(a) = 0 = g(b)$. Thus by *Rolle's Theorem* there is a point $c \in (a,b)$ so that $g'(c) = 0$. But, if $g'(c) = 0$, then $f'(c) = L'(c) = \frac{f(b) - f(a)}{b - a}$. ■

Corollary 1: *Let f be a differentiable function on (a,b) such that* $f'(x)=0$ *for all* $x\in(a,b)$*. Then f is a constant function on (a,b).*

Proof: If f is not constant, there exist $x_0,x_1\in(a,b)$ so that $a<x_0<x_1<b$ and $f(x_0)\neq f(x_1)$. By the *Mean Value Theorem*, for some $c\in(x_0,x_1)$ we have that

$$f'(c)=\frac{f(x_1)-f(x_0)}{x_1-x_0}\neq 0,$$

which is a contradiction. ■

Corollary 2: *Let f and g be differentiable functions on (a,b) such that* $f'=g'$ *for all* $x\in(a,b)$*. Then there is a constant C such that f (x) = g(x) + C for all* $x\in(a,b)$*.*

Proof: Left to the reader.

This is a very important result for integral calculus because it guarantees that all antiderivatives for a given function differ by at most a constant. Thus, when you have found one antiderivative, all the others will look like that one plus a constant. Note that the constant is not always explicit.

$$\begin{aligned}\int \sin x\cos x\,dx &= \frac{1}{2}\sin^2 x + C \text{ by substitution } u=\sin x\\ &= \frac{1}{2}\int \sin 2x\,dx \text{ using the trig identity } \sin 2x = 2\sin x\cos x\\ &= -\frac{1}{4}\cos(2x)+C\end{aligned}$$

Your job is to see how those two are the same.

Let f be a real-valued function defined on an interval I. We say that f is *strictly increasing on I* if $x_1,x_2\in I$ *and* $x_1<x_2$ *then* $f(x_1)<f(x_2)$; *strictly decreasing on I if* $x_1,x_2\in I$ *and* $x_1<x_2$ *then* $f(x_1)>f(x_2)$; *increasing on I if* $x_1,x_2\in I$ *and* $x_1<x_2$ *then* $f(x_1)\le f(x_2)$; and *decreasing on I* if $x_1,x_2\in I$ and $x_1<x_2$ then $f(x_1)\ge f(x_2)$.

Corollary 1: *Let f be a differentiable function on an interval* (*a,b*)*. Then*

i. *f is strictly increasing if* $f'(x)>0$ *for all* $x\in(a,b)$;
ii. *f is strictly decreasing if* $f'(x)<0$ *for all* $x\in(a,b)$;
iii. *f is increasing if* $f'(x)\ge 0$ *for all* $x\in(a,b)$;
iv. *f is decreasing if* $f'(x)\le 0$ *for all* $x\in(a,b)$.

Proof: We will show the proof for the first, as the others are proven similarly.

Consider x_0,x_1 where $a<x_0<x_1<b$. By the *Mean Value Theorem*, for some $c\in(x_0,x_1)$ we have

$$\frac{f(x_1)-f(x_0)}{x_1-x_0}=f'(c)>0.$$

Since $x_1-x_0>0$, we have that $f(x_1)-f(x_0)>0$ or $f(x_1)>f(x_0)$. ■

Intermediate Value Theorem for Derivatives: *Let f be a differentiable function on (a,b). Whenever* $a < x_0 < x_1 < b$ *and* m *lies between* $f'(x_0)$ *and* $f'(x_1)$, *then there is a* $c \in (x_1, x_2)$ *so that* $f'(c) = m$.

Proof: Without loss of generality we may assume that $f'(x_0) < m < f'(x_1)$. Let $g(x) = f(x) - mx$ for $x \in (a,b)$. Then $g'(x_0) < 0 < g'(x_1)$. We know that g assumes its minimum on $[x_0, x_1]$ at some point $c \in [x_0, x_1]$. Since $g'(x_0) = \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} < 0$, $g(x) - g(x_0) < 0$ for x close to and larger than x_0. In particular, there exists $y_0 \in (x_0, x_1)$ so that $g(y_0) < g(x_0)$. Therefore g does not assume its minimum at x_0, so we must have that $c \neq x_0$. Likewise, we can show that $c \neq x_1$, so that we have $c \in (x_0, x_1)$ and $g'(c) = 0$ by our previous theorem. Therefore, $f'(c) = g'(c) + m = m$. ■

VII. l'Hospital's Rule}

When we are computing limits, we often have to compute limits of the form

$$\lim_{x \to M} \frac{f(x)}{g(x)}$$

where *M* can signify a, a^+, a^-, $+\infty$, or $-\infty$. We know from our definition that the limit exists and

$$\lim_{x \to M} \frac{f(x)}{g(x)} = \frac{\lim_{x \to M} f(x)}{\lim_{x \to M} g(x)}$$

provided that $\lim_{x \to M} f(x)$ and $\lim_{x \to M} g(x)$ exist and are finite **and** provided that $\lim_{x \to M} g(x) \neq 0$. There are cases though where the definition does not help. When we arrive at an indeterminate form such as $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then we can apply *l'Hospital's Rule* to help us try to find the limit. Moreover, other indeterminate forms, such as $\infty - \infty$, 1^∞, ∞^0, 0^0 or $0 \times \infty$, can usually be reformulated so as to take the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

We need a few tools to prove l'Hospital's Rule. So first we prove the Generalized Mean Value Theorem.

Generalized Mean Value Theorem: *Let f and g be continuous functions on* $[a,b]$ *that are differentiable on* (a,b). *Then there exists at least one* $c \in (a,b)$ *such that*

$$f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)].$$

Proof: Define a function *h* on $[a,b]$ by

$$h(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)].$$

We have reduced the problem to showing that $h'(c) = 0$ for some $c \in (a,b)$. Now, h is continuous on $[a,b]$ and differentiable on (a,b). Now

$$h(a) = f(a)[g(b) - g(a)] - g(a)[f(b) - f(a)] = f(a)g(b) - g(a)f(b)$$

and

$$h(b) = f(b)[g(b) - g(a)] - g(b)[f(b) - f(a)] = -g(a)f(b) + f(a)g(b) = h(a).$$

Thus, *Rolle's Theorem* guarantees a $c \in (a,b)$ so that $h'(c) = 0$, and we are done. ■

Theorem 8: *Let M signify a,* a^+, a^-, $+\infty$, *or* $-\infty$, *where* $a \in \mathbb{R}$, *and suppose that f and g are differentiable functions for which the following limit exists:*

$$\lim_{x \to M} \frac{f'(x)}{g'(x)} = L. \qquad (1)$$

If

$$\lim_{x \to M} f(x) = \lim_{x \to M} g(x) = 0, \qquad (2)$$

or if

$$\lim_{x \to M} |g(x)| = +\infty, \qquad (3)$$

then

$$\lim_{x \to M} \frac{f(x)}{g(x)} = L. \qquad (4)$$

Note that the statement in Equation (1) means that we must assume that *f* and *g* must be defined and differentiable "near" *M* and $g'(x)$ must be *nonzero* "near" *M*.

Proof: The idea of the proof is to reduce the number of cases as much as possible to a few (one preferably) cases that are easily handled.

The case of $\lim_{x \to M} h(x)$ follows from the cases $\lim_{x \to M^+} h(x)$ and $\lim_{x \to M^-} h(x)$ since the former exists if and only if the latter two exist and are equal. We can restrict our attention to $\lim_{x \to M^+} h(x)$ and $\lim_{x \to -\infty}$, since the other cases are handled analogously. Finally, we can handle these cases together since they are very close to one another, as you would have to approach $-\infty$ from the right. More precisely, if L is finite, then $\lim_{x \to M^+} h(x) = L$ if and only if for each $\epsilon > 0$ there exists an $\alpha > M$ such that if $0 < x < \alpha$ then $|h(x) - L| < \epsilon$, since $\alpha > M$ if and only if $\alpha = M + \delta$ for some $\delta > 0$. If we set $M = -\infty$ above, we have exactly the same condition that verifies that $\lim_{x \to -\infty} h(x) = L$.

Thus, we may assume that $M \in \mathbb{R}$ or $M = -\infty$. We want to show that if $-\infty \leq L \leq \infty$ and $L_1 > L$, then there is $\alpha_1 > a$ so that

$$M < x < \alpha_1 \text{ implies } f(x)/g(x) < L_1. \qquad (5)$$

A similar argument would then show that if $-\infty \leq L \leq \infty$ and $L_2 < L$, then there is $\alpha_2 > a$ so that

$$M < x < \alpha_1 \text{ implies } f(x)/g(x) > L_2. \qquad (6)$$

We will prove the Equation (5) in a minute. First, let's use the Equations (5) and (6) to finish the proof. Assume that L is finite and let $\epsilon > 0$. Apply condition (5) to $L_1 = L + \epsilon$ and condition (6) to $L_2 = L - \epsilon$ to find $\alpha_1 > \alpha$ and $\alpha_2 > \alpha$ satisfying

$$M < x < \alpha_1 \quad \text{implies} \ \frac{f(x)}{g(x)} < L + \epsilon,$$

$$M < x < \alpha_2 \quad \text{implies} \ \frac{f(x)}{g(x)} > L - \epsilon.$$

Thus, if $\alpha = \min\{\alpha_1, \alpha_2\}$, then if $M < x < \alpha$ we have that

$$\left| \frac{f(x)}{g(x)} - L \right| < \epsilon;$$

which shows that

$$\lim_{x \to M^+} \frac{f(x)}{g(x)} = L.$$

If $L = -\infty$, then condition (5) and the fact that L_1 is arbitrary show that

$$\lim_{x \to M^+} \frac{f(x)}{g(x)} = -\infty.$$

If $L = +\infty$, then condition (6) and the fact that L_2 is arbitrary show that

$$\lim_{x \to M^+} \frac{f(x)}{g(x)} = \infty.$$

Now, we need to prove condition (5). Let $L_1 > L \geq -\infty$. We need to show that there is $\alpha_1 > M$ satisfying condition (5). Let (a,b) be an interval on which f and g are differentiable and on which g' never vanishes. Thus, g' is either always positive or always negative on (a,b). These can be handled together by replacing g by $-g$, so we may assume that $g' < 0$ for all $x \in (a,b)$. Thus, g is strictly decreasing on (a,b). Since g is one-to-one on (a,b), $g(x)$ can equal 0 for at most one x in (a,b). By choosing b smaller if necessary, we may assume that g never vanishes on (a,b). Now, choose K so that $L < K < L_1$. By (1) there is an $\alpha > M$ so that if $M < x < \alpha$ then

$$\frac{f'(x)}{g'(x)} < K.$$

If $M < x < y < \alpha$, then the *Generalized Mean Value Theorem* shows that

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(z)}{g'(z)}$$

for some $z \in (x,y)$.

Therefore, if $M < x < y < \alpha$, then

$$\frac{f(x) - f(y)}{g(x) - g(y)} < K. \qquad (7)$$

If (2) holds, then

$$\lim_{x\to M^+}\frac{f(x)-f(y)}{g(x)-g(y)}=\frac{f(y)}{g(y)},$$

so the above shows that

$$\frac{f(y)}{g(y)}\le K<L_1 \text{ for } M<y<\alpha;$$

in which case (5) holds.

Now, if (3) holds, then $\lim_{x\to M^+} g(x)=+\infty$ since g is strictly decreasing on (a,b). Also, $g(x)>0$ for $x\in(a,b)$, since g never vanishes on (a,b). We multiply both sides of (7) by $\frac{g(x)-g(y)}{g(x)}$, which is positive, we see that if $M<x<y<\alpha$ then

$$\frac{(f(x)-f(y)}{g(x)}<K\cdot\frac{g(x)-g(y)}{g(x)}$$

and therefore

$$\frac{f(x)}{g(x)}<K\cdot\frac{g(x)-g(y)}{g(x)}+\frac{f(y)}{g(x)}=K+\frac{f(y)-Kg(y)}{g(x)}.$$

Now, we may regard y as fixed and note that

$$\lim_{x\to M^+}\frac{f(y)-Kg(y)}{g(x)}=0.$$

Therefore, there is $\alpha_2>M$ so that $\alpha_1\le y<\alpha$ and if $M<x<\alpha_2$ then

$$\frac{f(x)}{g(x)}<L_1.$$

Thus, (5) holds. ■

VIII. Where l'Hospital's Rule Applies and Where it Doesn't

We need to remember the hypotheses of *l'Hospital's Rule* or we may run into trouble.

$$\lim_{x\to 0}\frac{\cos x-1}{x^2}=\lim_{x\to 0}\frac{-\sin x}{2x}=-\frac{1}{2}$$

$$\lim_{x\to\infty}\frac{x^3}{e^x}=\lim_{x\to\infty}\frac{3x^2}{e^x}=\lim_{x\to\infty}\frac{6x}{e^x}=\lim_{x\to\infty}\frac{6}{e^x}=0$$

Realizing that $\frac{d^n}{dx^n}x^n=n!$ gives us that $\lim_{x\to\infty}\frac{x^n}{e^x}=0$.

If we try the following:

$$\lim_{x\to 0^+}\frac{\log x}{x}=\lim_{x\to 0^+}\frac{1/x}{1}=+\infty,$$

we are wrong! Why? $\lim_{x\to 0^+}\log x = -\infty$ but $\lim_{x\to 0^+} x = 0$ so that neither (2) nor (3) in our theorem is satisfies. Instead, we rewrite our fraction so that the conditions are met.

$$\log x = -\log\left(\frac{1}{x}\right).$$

Thus,

$$\begin{aligned}\lim_{x\to 0^+}\frac{\log x}{x} &= \lim_{x\to 0^+} -\frac{\log(1/x)}{x} \\ &= \lim_{x\to 0^+} -\frac{1}{x}\log(1/x) \\ &= \lim_{y\to\infty} -y\log y = -\infty\end{aligned}$$

Example 1: Consider

$$\lim_{x\to 0^+} x\log x = 0\cdot(-\infty).$$

We need to rewrite this to apply l'Hospital's Rule. If we rewrite

$$x\log x = \frac{\log x}{1/x},$$

Then we get the indeterminate form $-\infty/\infty$, so we can apply l'Hospital's Rule:

$$\lim_{x\to 0^+} x\log x = \lim_{x\to 0^+}\frac{\log x}{1/x} = \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = -\lim_{x\to 0^+} x = 0.$$

Example 2: Consider

$$\lim_{x\to 0^+} x^x = 0^0.$$

Rewrite the form:

$$x^x = e^{x\log x},$$

so

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\log x} = e^0 = 1.$$

Example 3: $\lim_{x\to\infty} x^{1/x} = \infty^0$.

Rewrite the form:

$$x^{1/x} = e^{\log x/x}.$$

By l'Hospital's Rule

$$\lim_{x\to\infty}\frac{\log x}{x} = \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = 0.$$

Therefore,

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\log x/x} = e^0 = 1.$$

Example 4: $\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x = 1^\infty$.

Rewrite the form:

$$\left(1-\frac{1}{x}\right)^x = e^{x\log(1-1/x)}.$$

By l'Hospital's Rule

$$\lim_{x\to\infty} x\log\left(1-\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\log\left(1-\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\left(1-\frac{1}{x}\right)^{-1}x^{-2}}{x^{-2}}$$

$$= \lim_{x\to\infty} -\left(1-\frac{1}{x}\right)^{-1} = -1$$

Therefore,

$$\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x = e^{-1}.$$

Example 5: Consider the limit $\lim_{x\to 0} h(x)$ where

$$h(x) = \frac{1}{e^x-1} - \frac{1}{x} \text{ for } x \neq 0.$$

Neither of the limits

$$\lim_{x\to 0}(e^x-1)^{-1} \text{ nor } \lim_{x\to 0} x^{-1}$$

exists, so we are not in the right form, as written. However, $\lim_{x\to 0^+} h(x)$ is an indeterminate form of the form $\infty-\infty$ and $\lim_{x\to 0^-} h(x)$ is an indeterminate form of the form $(-\infty)-(-\infty)$. So we need to rewrite the expression. Add the fractions and

$$h(x) = \frac{1}{e^x-1} - \frac{1}{x} = \frac{x-e^x+1}{x(e^x-1)}.$$

This limit is an indeterminate form of the form $\frac{0}{0}$ so by l'Hospital's Rule,

$$\lim_{x\to 0}\frac{x-e^x+1}{x(e^x-1)} = \lim_{x\to 0}\frac{1-e^x}{xe^x+e^x-1} = \lim_{x\to 0}\frac{-e^x}{xe^x+2e^x} = -\frac{1}{2}.$$

Note that $xe^x+2e^x \neq 0$ for $x\in(-2,\infty)$, so our result holds.

Example 6: Consider $f(x) = x^2\sin\left(\frac{1}{x}\right)$ and $g(x) = \sin x$. Look at

$$\lim_{x\to 0}\frac{f(x)}{g(x)}.$$

Both the numerator and denominator go to 0 as $x \to 0$. However,

$$f'(x) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

$$g'(x) = \cos x$$

Note that $\lim_{x\to 0} f'(x)$ does not exist, and thus neither does $\lim_{x\to 0} f'(x)/g'(x)$. Hence, l'Hospital's Rule does not apply!!

We can find the limit, however:

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{x^2\sin\left(\frac{1}{x}\right)}{\sin x} = \lim_{x\to 0}\frac{x}{\sin x}\cdot x\sin\left(\frac{1}{x}\right) = 1\cdot 0 = 0.$$

Example 7: Let $f(x) = \frac{\sin x}{x^2}$ and $\frac{1}{x}$.

Then,

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{\sin x}{x} = 0.$$

If we were to look at it as the indeterminate form 0/0 and try to apply l'Hospital's Rule, we would get:

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{\frac{\sin x}{x^2}}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\frac{x^2\cos x - 2x\sin x}{x^4}}{-\frac{1}{x^2}} = \lim_{x\to\infty}\frac{2\sin x}{x} - \cos x$$

which does not exist.

Example 8: We have similar problems in the following. The original limits would indicate that you should apply l'Hospital's Rule but the "derived" limits do not exist, while the original rational limit does exist.

$$f(x) = x\sin^2 x \qquad f'(x) = \sin^2 x + 2x\sin x\cos x$$

$$g(x) = x^2$$

$$a = \pm\infty$$

$$\frac{f'(x)}{g'(x)} = \frac{\sin 2x}{2} + \frac{\sin^2 x}{2x}$$

So, $\lim_{x\to 0}\frac{f'(x)}{g'(x)}$ does not exist!

$$\frac{f(x)}{g(x)} = \frac{\sin^2 x}{x}$$

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = 0$$

Thus, it is important to remember that you can apply l'Hospital's Rule ***only*** when $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists. We have seen in this last example that it is possible for $\lim_{x\to a} f(x)/g(x)$ to exist but $\lim_{x\to a} f'(x)/g'(x)$ does not exist.