

Families of Functions

One very important area of real analysis is the study of the properties of real valued functions. We introduced the general properties of functions earlier in Chapter 6. In this chapter we want to look at specific functions and families of functions to see what properties really belong to the area of analysis separate from algebra or topology. We will take on the functions in a somewhat orderly manner.

Polynomials

The polynomials in the study of functions play a similar role to the integers in our study of numbers. They are our basic building blocks. Likewise, when we really start looking at the family of polynomials most of the questions we have are really algebraic in nature and not analytic, just like the study of integers.

In general a polynomial is an expression in which a finite number of constants and variables are combined using only addition, subtraction, multiplication, and nonnegative whole number exponents (raising to a power).

A polynomial function is a function defined by evaluating a polynomial. For example, the function f defined by $f(x) = x^3 - x$ is a polynomial function. Polynomial functions are an important class of *smooth* functions — smooth meaning that they are infinitely differentiable, *i.e.*, they have derivatives of all finite orders. Because of their simple structure, polynomials are easy to evaluate, and are used extensively in numerical analysis for polynomial interpolation or to numerically integrate more complex functions. Computationally, they require only multiplication and addition. Any division is division of coefficients and subtraction is just the addition of the additive inverse.

Polynomials have a number of nice properties.

- A sum of polynomials is a polynomial.
- A product of polynomials is a polynomial.
- The derivative of a polynomial is a polynomial.
- The integral of a polynomial is a polynomial

Polynomials are used to approximate other functions, such as sine, cosine, and the exponential and logarithmic functions. We will characterize those functions that can be approximated by polynomials later.

All polynomials have an expanded form, in which the distributive law has been used to remove all parentheses. (Some polynomials also have a factored form, in which parentheses appear.) In expanded form, a term of a polynomial is a part of the polynomial that is the product of a number (called the coefficient) and zero or more variables, where a variable that appears more than once is expressed as a power of that variable. Every polynomial in expanded form is a sum of terms, where subtraction is carried out by addition of terms with negative coefficients.

Every polynomial in one variable is equivalent to a polynomial with the form $a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$.

This form is sometimes taken as the definition of a polynomial in one variable.

A polynomial equation is an equation in the form of a polynomial equal to zero or equal to another polynomial. The latter form is usually converted to the former by subtracting the second polynomial from both sides of the equation. When written as a polynomial equal to zero, the degree of the equation is the degree of the polynomial.

In elementary algebra, methods are given for solving all first degree and second degree polynomial equations in one unknown. The number of solutions may equal the degree, but it is necessary to consider multiplicity of solutions and complex numbers for this to be universally true, as described in the Fundamental Theorem of Algebra.

Determining the roots of polynomials, or "solving algebraic equations", is one of the oldest problems in mathematics. Some polynomials, such as $f(x) = x^2 + 1$, do not have any real roots. If, however, the set of allowed candidates is expanded to the complex numbers, every (non-constant) polynomial has a root | in fact it has n roots, counting multiplicity, if the degree of the polynomial is n . This is the statement of the Fundamental Theorem of Algebra.

Polynomial functions

For given numerical constants a_0, \dots, a_n , $n > 0$, in some field (often $\mathbb{Q}; \mathbb{R}$ or \mathbb{C}), with a_n non-zero, then a polynomial (function) of degree n is a function of the form

$$f(x) = a_0 + a_1x + \dots + a_{n-1}x^{n-1} + a_nx^n$$

More concisely, a polynomial function can be written in sigma notation as

$$f(x) = \sum_{k=0}^n a_k x^k$$

The constants a_0, \dots, a_n are called the coefficients of the polynomial. a_0 is called the constant coefficient and a_n is called the leading coefficient. When the leading coefficient is 1, the polynomial is said to be *monic*. Each summand $a_k x^k$ of the polynomial is called a *term*. A polynomial with one, two or three terms is called monomial, binomial or trinomial respectively.

Graphs

A polynomial function in one real variable can be represented by a graph.

- The graph of the zero polynomial $f(x) = 0$ is the x -axis.
- The graph of a degree 0 polynomial $f(x) = a_0$, where $a_0 \neq 0$, is a horizontal line with y -intercept a_0 .
- The graph of a degree 1 polynomial (or linear function) $f(x) = a_0 + a_1x$, where $a_1 \neq 0$, is an line with y -intercept a_0 and slope a_1 .
- The graph of a degree 2 or higher polynomial $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$,

where $a_n \neq 0$ and $n \geq 2$ is a continuous non-linear curve.

The best way to analyze the graph of a degree 2 or higher polynomial function is by its end behavior, the number of x -intercepts and the number of turning points.

End behavior

There are four end behaviors which are direct results of whether a_n , the leading coefficient, is positive or negative and whether n , the degree of the polynomial, is even or odd.

1. If a_n is positive and n is even, the right end of the polynomial is in quadrant I while the left end is in quadrant II.
2. If a_n is negative and n is even, the right end is in quadrant IV while the left end is in quadrant III.
3. If a_n is positive and n is odd, the right end is in quadrant I while the left end is in quadrant III.
4. If a_n is negative and n is odd, the right end is in quadrant IV while the left end is in quadrant II.

Number of x -intercepts

From the Fundamental Theorem of Algebra, a polynomial of degree n has exactly n complex roots, which may or may not be real. Therefore, the number of x -intercepts cannot exceed n . It also follows from the Fundamental Theorem of Algebra that the complex roots of a polynomial with real coefficients must exist in complex-conjugate pairs. This implies that an even-degree polynomial with real coefficients may have no x -intercepts (because all its roots may be complex) and that an odd-degree polynomial with real coefficients must have at least one x -intercept, since any pairing of roots into complex-conjugate pairs will necessarily leave at least one root unpaired for odd n . This (or these) *unpaired* root(s) must therefore be real.

Number of turning points

The number of turning points of an even-degree polynomial is any odd number less than the degree, while the number of turning points of an odd-degree polynomial is any even number less than the degree. For example, a degree 4 polynomial function can have 1 or 3 turning points whereas a degree 5 polynomial function can have 0, 2, or 4 turning points.

On your graphing calculator, graph the following examples of polynomials of low degree.

$$1. f(x) = x^2 - x - 2 = (x+1)(x-2)$$

$$2. f(x) = \frac{x^3}{5} + \frac{4x^2}{5} - \frac{7x}{5} - 2 = \frac{1}{5}(x+5)(x+1)(x-2)$$

$$3. f(x) = \frac{1}{14}(x+4)(x+1)(x-1)(x-3) + 0.5$$

$$4. f(x) = \frac{1}{20}(x+4)(x+2)(x+1)(x-1)(x-3) + 2$$

$$5. f(x) = 13x^4 - 7x^3 + \frac{2}{3}x^2 - 5x + 3.$$

Roots

A *root* or *zero* of a polynomial p is a number r so that $p(r) = 0$. The Fundamental Theorem of Algebra states that a polynomial of degree n over the complex numbers has exactly n complex roots (not necessarily distinct ones). Therefore a polynomial can be factored as

$$p(z) = a_n(z - r_1)(z - r_2)\dots(z - r_n),$$

where each r_i is a root of the polynomial p . For example, the polynomial $x^3 - x$ can be factored as $x(x-1)(x+1)$. An equation of the form $p(x) = 0$ is called a polynomial equation in x . The solutions to the equation are the roots of the polynomial. The real roots of a polynomial (with real coefficients) are the x -intercepts of the graph of $p(x)$ viewed as a polynomial function.

One important aspect of calculus is the project of analyzing complicated functions by means of approximating them with polynomials. The culmination of these efforts is Taylor's theorem, which roughly states that every differentiable function locally looks like a polynomial, and the Stone-Weierstrass theorem, which states that every continuous function defined on a compact interval of the real axis can be approximated on the whole interval as closely as desired by a polynomial. Polynomials are also frequently used to interpolate functions.

Rational Functions

A rational function is a function that is the quotient of two polynomial functions.

$$f(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomial functions, and $q(x)$ is not identically zero. Note that there is no condition that states $p(x)$ and $q(x)$ have no factors in common. In fact, you must realize that the two functions

$$f(x) = \frac{x^2 - 1}{x - 1} \text{ and } g(x) = x + 1$$

are different as functions, though they are algebraically identical. Note that the domain of f does not contain 1, while 1 is in the domain of g . Usually we will consider the rational functions whose numerator and denominator have no factors in common, because it will give us the same function just with a larger domain.

Example: Let

$$f(x) = \frac{x^2 + x - 20}{x^2 - 3x - 18}.$$

To understand the behavior of this rational function it is useful to see its constituent polynomials in factored form. The polynomials in the numerator and the denominator of the above function would factor like this:

$$f(x) = \frac{(x+5)(x-4)}{(x+3)(x-6)}.$$

The Domain

By considering the constituent polynomials in factored form, the roots of the denominator are obvious. In the above example they are $x = -3$ and $x = 6$. That is, if x takes on either of these two values, the denominator becomes equal to zero. Since we cannot divide by zero, the function is not defined for these two values of x . We say that the function is not defined at $x = -3$ and $x = 6$.

Other values for x do not cause the function to become undefined, so, we say that the function is defined for all other values for x . In other words, all real numbers except -3 and 6 are allowed as inputs to this function. The domain for the function therefore is:

$$\{x \in \mathbb{R} \mid x \neq -3 \text{ or } 6\} = (-\infty, -3) \cup (-3, 6) \cup (6, \infty).$$

The x -intercepts

The x -intercepts for this function would be where the output, or y -value, equals zero. A rational function is a fraction, and a fraction is equal to zero when the numerator is equal to zero. For the above rational function this happens at the roots of this polynomial. The roots of the numerator polynomial are $x = -5$ and $x = 4$. That is, when x takes on either of these two values the numerator becomes zero, and the output of the function, or y -value, also becomes zero.

Notice that the function must be defined at these values. In our other example

$$g(x) = \frac{x^2 - 1}{x - 1}$$

even though the numerator is zero at $x = 1$, the function is not defined there. Thus, $x = 1$ cannot be an x -intercept.

The y -intercept

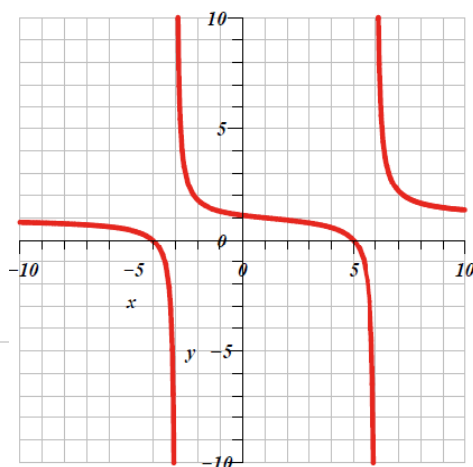
What about the y -intercept? Of course, this is where $x = 0$. As a matter of ease, it is often easier for students to use the unfactored form when computing the y -intercept.

In the example above

$$f(0) = \frac{0^2 + 0 - 20}{0^2 - 3 \cdot 0 - 18} = \frac{-20}{-18} = \frac{10}{9}.$$

The Graph

Consider a graph of the function in the figure below. The graph of the function breaks at $x = -3$ and at $x = 6$. The function crosses the x -axis at $x = -5$ and $x = 4$. These are the same as the



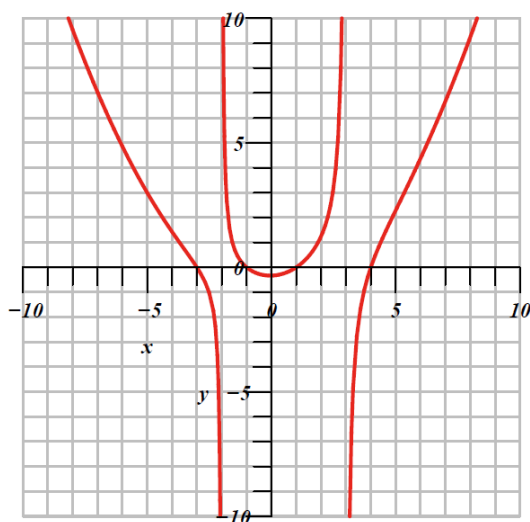
-3
at x

values which we calculated above for the x -intercepts. The function crosses the y -axis just a bit above 1, at about 1.1. This is the same location as the calculated y -intercept above.

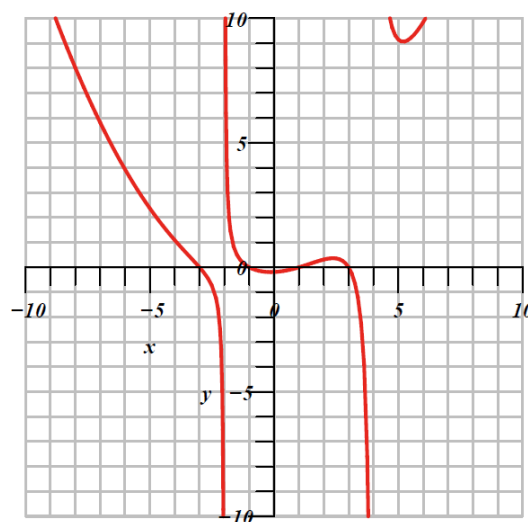
End Behavior

The way that a function is shaped when the input approaches positive or negative infinity is called its *end behavior*. One might say that the end behavior of a function describes what the function looks like from far away.

In our graph notice how the graph is leveling off to a height of about $y = 1$ as x gets large positively or negatively. That is, at the left and right edge of the graph the function starts to act like the function described $f(x) = 1$. What would you consider to be the end behavior of the following two functions?



$$\frac{x^4 - 7x^3 + 11x^2 + 7x - 12}{5x^2 - 5x - 30}$$



$$\frac{x^4 - 10x^2 + 9}{5x^2 - 10x - 40}$$

The Range

An examination of the graph above will indicate that all real numbers are available for output. The left most fork of the graph looks like it will rise not quite to $y = 1$, and the right most fork looks like it may not quite drop to $y = 1$; so, one might at first suggest that $y = 1$ is not in the range. However, if you examine the central portion of the graph, you can see that $y = 1$ is definitely a member of the range. So, all real numbers are present in the range.

Algebraic Functions

A function which can be constructed using only a finite number of elementary operations — addition, subtraction, multiplication, division, and exponentiation — together with the inverses of functions capable of being so constructed is called an algebraic function. This class of functions contains polynomial functions and rational functions. It includes more though, since it includes the root functions, such as the square root and the cube root. Examples of algebraic functions not previously considered would be functions like:

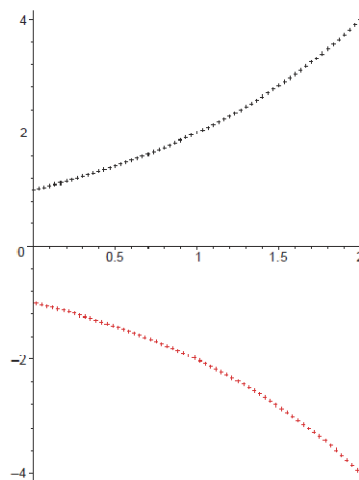
$$f(x) = \sqrt{ax^3 + bx^2 + cx + d}$$

$$g(x) = \sqrt[3]{3x^4 - 8}$$

$$h(x) = ax^{1/2} + bx^{1/3} + cx^{1/4}$$

Functions involving expressions such as $\sqrt{x^2 + a^2}$, $\sqrt{x^2 - a^2}$, $\sqrt{a^2 - x^2}$, $\sqrt{ax + b}$ and $\sqrt{ax^2 + bx + c}$ are sometimes called *irrational functions*. These expressions often arise in calculus in antidifferentiation problems. Long lists of antiderivatives exist for them in multiple places. These, too, are algebraic functions.

Note that it is also possible to talk about an algebraic function that is implicitly defined by an algebraic equation in 2 variables. You might not be able to have an explicit representation of the function, but the values of the function, $y(x)$, are determined by the values of x . This is not essentially different from the above definition, but lends itself to generalizing to multiple variables.



Transcendental Functions

A function which is not algebraic is a transcendental function. This includes the exponential, logarithmic, and trigonometric functions — which we will discuss later. It also includes many, many more functions called special functions. They are called this because they normally arise in special situations and are not as widely used as others. We will look at a few of these: the Lambert W function and glog . Many others exist that can be found in most Differential Equations and Numerical Analysis texts. A very important part of Real Analysis is finding ways to study and describe all functions, and this includes special functions.

Exponential Functions

We are used to functions where the variable is raised to a fixed power. The exponential function reverses the roles of the base and the exponent in power functions. To carefully define the exponential function we will need a number of things that we have not yet proven or derived

A function of the form $f(x) = a^x$ for $a \neq 1$ and $a > 0$ is called an exponential function. If we were to take $a = 1$ then this is just the constant function, which is already contained in the set of polynomials. If we take $a < 0$, then the function is not defined for all real numbers. For example, if we consider the expression $(-2)^x$ and graph this over the interval from 0 to 2 we might get a graph like the figure on the right.

e^x is always strictly positive for any x , so the domain of the function is the set of real numbers and the codomain is the set of positive real numbers. The exponential function obeys all of the normal laws of exponents, *i.e.*

$$(a) \quad e^x e^y = e^{x+y}.$$

Figure 1: $(-2)^x$ for carefully chosen values of $x \in [0, 2]$

(b) $e^{x+y} = e^x e^y.$

(c) $e^{-x} = \frac{1}{e^x}$

(d) $e^{x-y} = \frac{e^x}{e^y}.$

In fact, we could start with the condition that $f(x+y) = f(x)f(y)$. Notice then that this makes $f(0) = [f(0)]^2$ so either $f(0) = 0$ or $f(0) = 1$. Since $f(x) = f(0+x) = f(0)f(x)$, if $f(0) = 0$, then we would have that $f(x) = 0$ for all x and this is not that interesting. Therefore, $f(0) = 1$. Note that this now implies that $f(-x) = 1/f(x)$, since

$$f(-x)f(x) = f(-x+x) = f(0) = 1.$$

It then follows that $f(x-y) = f(x)/f(y)$.

Now, let $f(1) = \alpha$. Then for any positive integer n , $f(n) = \alpha^n$ by induction. Then we can see that $f(-n) = 1/\alpha^n = \alpha^{-n}$. Let r be a rational number and let $r = \frac{p}{q}$. Then

$$\alpha^p = f(p) = f\left(q \cdot \frac{p}{q}\right) = f\left(\frac{p}{q}\right)^q.$$

From this it follows that $f(p/q) = \alpha^{p/q}$. Now, if we knew that f were continuous, we could then show that $f(x) = \alpha^x$ for all real numbers x . This tells us that any continuous, real-valued function that satisfies $f(x+y) = f(x)f(y)$ must be an exponential function. What kind of function would satisfy $f(xy) = f(x) + f(y)$?

What can we say about α ? We know that if $\alpha = 1$, then $f(x) = 1$ and this would be a constant function. If we want this to be a real-valued function, then since we have shown that $f\left(\frac{1}{2}\right) = \sqrt{\alpha}$, we would want to keep α from being negative. If we were to allow complex values as output, then that restriction need not be applied. Also, if we were to have $\alpha = 0$, then $f(x)$ would be undefined for all $x < 0$. Thus, we will have the restrictions that $\alpha > 0$ and $\alpha \neq 1$.

Is this function a one-to-one function? What happens if $f(x) = f(y)$?

$$f(x) = f(y)$$

$$\alpha^x = \alpha^y$$

$$\alpha^{x-y} = 1$$

$$x - y = 0$$

$$x = y$$

Thus, the exponential function is one-to-one.

Logarithmic Functions

Since the exponential function is one-to-one, it then has an inverse function defined for $x > 0$ since the range of the exponential function is all positive reals. The logarithm is the mathematical operation that is the inverse of exponentiation (raising a constant, the base, to a power). The logarithm of a number x in base b is the number n such that $x = b^n$. Thus, we have that

$$\log_b(x) = n \iff b^n = x.$$

Logarithms reduce multiplication operations to addition, division to subtraction, exponentiation to multiplication, and roots to division. Therefore, logarithms are useful for making lengthy numerical operations easier to perform and, before the advent of electronic computers, they were widely used for this purpose in fields such as astronomy, engineering, navigation, and cartography. They have important mathematical properties and are still used in many ways.

The most widely used bases for logarithms are 10, 2 and the Euler constant $e \sim 2.718281828459045\dots$. When \log is written without a base, *i.e.* the b is missing from \log_b , the intent can usually be determined from context:

- natural logarithm (\log_e) in mathematical analysis,
- common logarithm (\log_{10}) in engineering and when logarithm tables are used to simplify hand calculations,
- binary logarithm (\log_2) in information theory¹ and musical intervals

The notation $\ln(x)$ almost always means $\log_e(x)$, *i.e.*, the natural logarithm of x , but the implied base for $\log(x)$ varies by discipline:

- Mathematicians generally understand both $\ln(x)$ and $\log(x)$ to mean $\log_e(x)$ and write $\log_{10}(x)$ when the base-10 logarithm of x is intended. Sometimes the term $\lg(x)$ is used for the base-2 logarithm of x .
- Engineers, biologists, and some others write only $\ln(x)$ or $\log_e(x)$ when they mean the natural logarithm of x , and take $\log(x)$ to mean $\log_{10}(x)$ or, sometimes in the context of computing, $\log_2(x)$.
- On most calculators, the LOG button is $\log_{10}(x)$ and LN is $\log_e(x)$. However, in most commonly used computer programming languages, including C, C++, Java, Fortran, and BASIC, the log function returns the natural logarithm. The base-10 function, if it is available, is generally \log_{10} .
- Some people use $\text{Log}(x)$ (capital L) to mean $\log_{10}(x)$, and use $\log(x)$ with a lowercase l to mean $\log_e(x)$.

¹ Information theory is a discipline in applied mathematics involving the quantification of data with the goal of enabling as much data as possible to be reliably stored on a medium and/or communicated over a channel. The measure of data, known as information entropy, is usually expressed by the average number of bits needed for storage or communication. Applications of fundamental topics of information theory include ZIP files (lossless data compression), MP3s (lossy data compression), and DSL (channel coding). The field is at the crossroads of mathematics, statistics, computer science, physics and electrical engineering, and its impact has been crucial to success of the Voyager missions to deep space, the invention of the CD, the feasibility of mobile phones, the development of the Internet, the study of linguistics and of human perception, the understanding of black holes, and numerous other fields.

- The notation $\text{Log}(x)$ is also used by mathematicians to denote the principal branch of the (natural) logarithm function.
- Also frequently used is the notation $\text{blog}(x)$ instead of $\log_b(x)$.

This chaos, historically, originates from the fact that the natural logarithm has nice mathematical properties (such as its derivative being $1/x$, and having a simple definition), while, back when logarithms were used to speed computations, the decimal logarithms were far better for that, so natural logarithms were unknown outside calculus classes while decimal logarithms were widely used in a variety of disciplines.

In computer science, the base 2 logarithm is sometimes written as $\lg(x)$ to avoid confusion. This usage was suggested by Edward Reingold and popularized by Donald Knuth. However, in Russian literature, the notation $\lg(x)$ is generally used for the base 10 logarithm, so even this usage is not without its perils.

While there are several useful identities, the most important for calculator use lets one find logarithms with bases other than those built into the calculator (usually \log_e and \log_{10}). To find a logarithm with base b using any other base a :

$$n = \log_b x \iff b^n = x$$

$$\log_a(b^n) = \log_a x$$

$$n \cdot \log_a b = \log_a x$$

$$n = \frac{\log_a x}{\log_a b}$$

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Logarithms are useful in solving equations in which exponents are unknown. They have simple derivatives, so they are often used in the solution of integrals. The logarithm is one of three closely related functions. In the equation $b^n = x$, b can be determined with radicals, n with logarithms, and x with exponentials.

Various quantities in science are expressed as logarithms of other quantities.

1. The negative of the base-10 logarithm is used in chemistry, where it expresses the concentration of hydronium ions (pH). The concentration of hydronium ions in neutral water is 10^{-7} at 25°C , hence a pH of 7.
2. The bel (symbol B) is a unit of measure which is the base-10 logarithm of ratios, such as power levels and voltage levels. It is mostly used in telecommunication, electronics, and acoustics. It is used, in part, because the ear responds logarithmically to acoustic power. Invented by engineers of the Bell Telephone Laboratory to quantify the reduction in audio level over a 1 mile (1.6 km) length of standard telephone cable, it was originally called the transmission unit or TU , but was renamed in 1923 or 1924 in honor of the laboratory's founder and telecommunications pioneer Alexander Graham Bell. The bel was too large for everyday use, so the decibel (dB), equal to 0.1 bel (B), became more commonly used.

3. The neper is a similar unit which uses the natural logarithm of a ratio. It is not an international system unit but is accepted for use alongside that system. It is used to express ratios, such as gain to loss and voltage and current, and relative values. The value of a ratio in nepers, Np , is given by

$$Np = \ln \frac{x_1}{x_2} = \ln x_1 - \ln x_2 .$$

4. The Richter scale measures earthquake intensity on a base-10 logarithmic scale.
5. In spectrometry and optics, the absorbance unit used to measure optical density is equivalent to $-1 B$.
6. In astronomy, the apparent magnitude measures the brightness of stars logarithmically, since the eye also responds logarithmically to brightness.

The discovery of logarithms just before Newton's era had an impact in the scientific world which can be compared with the invention of the computer in the 20th century, because many calculations which were too laborious became feasible.

When the chronometer was invented in the 18th century, logarithms allowed all calculations needed for astronomical navigation to be reduced to just additions, speeding the process by one or two orders or magnitude. A table of logarithms with five decimals, plus logarithms of trigonometric functions, was enough for most astronomical navigation calculations, and those tables fit in a small book.

To compute powers or roots of a number, the common logarithm of that number was looked up and multiplied or divided by the radix. Interpolation could be used for still higher precision. Slide rules used logarithms to perform the same operations more rapidly, but with much less precision than using tables. Other tools for performing multiplications before the invention of the calculator include Napier's bones and mechanical calculators.

History of Logarithms

In the 17th century, Joost Bürgi, a Swiss clockmaker employed by the Duke of Hesse-Kassel, first discovered logarithms as a computational tool. He did not publish his discovery, however, until 1620. The method of logarithms was first publicly discussed in 1614, in a book entitled *Mirifici Logarithmorum Canonis Descriptio*, by John Napier, Baron of Merchiston in Scotland, four years after the publication of his memorable discovery. This method contributed to the advance of science, and especially of astronomy, by making some difficult calculations possible. Prior to the advent of calculators and computers, it was used constantly in surveying, navigation, and other branches of practical mathematics. It supplanted the more involved *prosthaphaeresis*, which relied on trigonometric identities, as a quick method of computing products. Besides their usefulness in computation, logarithms also filled an important place in the higher theoretical mathematics.

At first, Napier called logarithms *artificial numbers* and antilogarithms *natural numbers*. Later, Napier formed the word *logarithm* to mean a number that indicates a ratio from the Greek *λογος* (logos) meaning proportion, and *αριθμος* (arithmos) meaning number. Napier chose that because the difference of two logarithms determines the ratio of the numbers for which they stand, so that an arithmetic series of

logarithms corresponds to a geometric series of numbers. The term antilogarithm was introduced in the late 17th century and, while never used extensively in mathematics, persisted in collections of tables until they fell out of favor with the advent of the electronic calculator.

Napier did not use a base as we now understand it, but his logarithms were, up to a scaling factor, effectively to base $1/e$. For interpolation purposes and ease of calculation, it is useful to make the ratio r in the geometric series close to 1.

Napier chose $r = 1 - 10^{-7} = 0.999999$, and Bürgi chose $r = 1 + 10^{-4} = 1.0001$. Napier's original logarithms did not have $\log 1 = 0$ but rather $\log 10^7 = 0$. Thus if N is a number and L is its logarithm as calculated by Napier, $N = 10^7(1 - 10^{-7})^L$. Since $(1 - 10^{-7})^{10^7}$ is approximately $1/e$, this makes $L/10^7$ approximately equal to $\log_{1/e} N/10^7$.

Prior to the advent of computers and calculators, using logarithms meant using tables of logarithms, that had to be created manually. Base-10 logarithms are useful in computations when electronic means are not available.

In 1617, Henry Briggs published the first installment of his own table of common logarithms, containing the logarithms of all integers below 1000 to eight decimal places. This he followed, in 1624, by his *Arithmetica Logarithmica*, containing the logarithms of all integers from 1 to 20,000 and from 90,000 to 100,000 to fourteen places of decimals, together with a learned introduction, in which the theory and use of logarithms are fully developed. The interval from 20,000 to 90,000 was filled up by Adriaan Vlacq, a Dutch computer; but in his table, which appeared in 1628, the logarithms were given to only ten places of decimals.

Vlacq's table was later found to contain 603 errors, but that was not regarded as a great number of errors "*when it is considered that the table was the result of an original calculation, and that more than 2,100,000 printed figures are liable to error.*"²

An edition of Vlacq's work, containing many corrections, was issued at Leipzig in 1794 under the title *Thesaurus Logarithmorum Completus* by Jurij Vega. Callet's seven-place table in 1795 gave the eight-place logarithms of the numbers between 100,000 and 108,000, in order to diminish the errors of interpolation, which were greatest in the early part of the table. The only important published extension of Vlacq's table was made by Mr. Sang 1871, whose table contained the seven-place logarithms of all numbers below 200,000.

Briggs and Vlacq also published original tables of the logarithms of the trigonometric functions. Besides the tables mentioned above, a great collection, called Tables du Cadastre, was constructed under the direction of Gaspard de Prony, by an original computation, under the auspices of the French republican government of the 1700's. The work contained the logarithms of all numbers up to 100,000 to nineteen places, and of the numbers between 100,000 and 200,000 to twenty-four places. It exists only in manuscript, however, "in seventeen enormous folios," at the Observatory of Paris. It was begun in 1792 and was completed in two years.

² Athenaeum, 15 June 1872. See also the Monthly Notices of the Royal Astronomical Society for May 1872.

Logarithms *via* Calculus

Consider the curve $y = 1/t$. We are interested in the area under the curve from $t = 1$ to $t = x$. Let $L(x) = \int_1^x 1/t dt$.

note that $L(1)=0$.

- 1) What is the area?
- 2) How is the area from 1 to ab related to the area from 1 to a and the area from 1 to b ?
- 3) What about a/b ?
- 4) What about a^b ?

We will assume that $1 < a < b$. Therefore, $a < ab$ and $b < ab$.

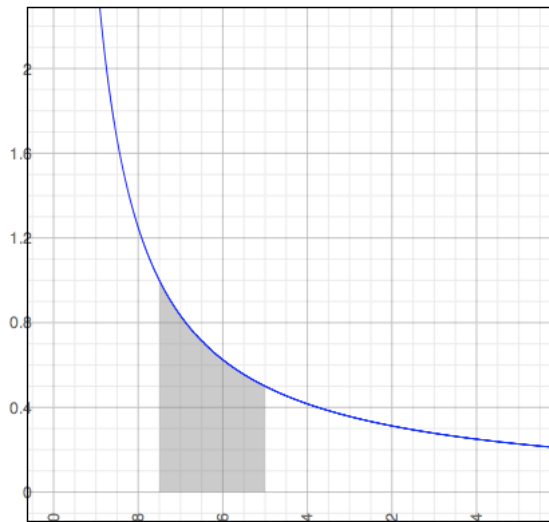
Lemma: For $1 < a < b$ we have

$$\int_a^{ab} \frac{1}{t} dt = \int_1^b \frac{1}{t} dt.$$

Proof: This is most easily done the simple substitution $x = t/a$.

$$\int_a^{ab} \frac{dt}{t} = \int_a^{ab} \frac{1/a dt}{t/a} = \int_1^b \frac{dx}{x} = L(b).$$

is what we were trying to prove.



First

a

with

This

From this we have the following lemma.

Lemma: $L(ab) = L(a) + L(b)$.

Proof:

$$\begin{aligned} L(ab) &= \int_1^{ab} \frac{1}{t} dt \\ &= \int_1^a \frac{1}{t} dt + \int_a^{ab} \frac{1}{t} dt \\ &= \int_1^a \frac{1}{t} dt + \int_1^b \frac{1}{t} dt \\ &= L(a) + L(b) \end{aligned}$$

Lemma: $L(1/b) = -L(b)$.

Proof: Assume $b > 1$.

$$L(1/b) = \int_1^{1/b} \frac{1}{t} dt = - \int_{1/b}^1 \frac{1}{t} dt = - \int_1^b \frac{1}{t} dt = -L(b)$$

From this it follows that $L(a/b) = L(a) - L(b)$.

Now, we need to consider $L(a^b)$. Let's look at $L(a^2)$ to start. Again, we will assume $a > 1$.

$$\begin{aligned}
 L(a^2) &= \int_1^{a^2} \frac{1}{t} dt = \int_1^a \frac{1}{t} dt + \int_a^{a^2} \frac{1}{t} dt \\
 &= \int_1^a \frac{1}{t} dt + \int_1^a \frac{1}{t} dt = 2L(a)
 \end{aligned}$$

This will extend by induction to $L(a^n) = nL(a)$, for any integer n . This will then extend to $L(a^r) = rL(a)$, for any rational number r . How do we extend this to any real number, b ? Since we were able to handle the previous cases with a simple substitution, maybe we can do the same here.

$$L(a^b) = \int_1^{a^b} \frac{dt}{t}$$

$$\text{Let } t = x^b$$

$$dt = bx^{b-1} dx$$

$$L(a^b) = \int_1^a \frac{bx^{b-1} dx}{x^b} = b \int_1^a \frac{dx}{x} = bL(a)$$

Therefore, we have a function $L(x)$ that is defined on $(0, \infty)$ that satisfies:

1. $L(1) = 0$,
2. $L(ab) = L(a) + L(b)$,
3. $L(a/b) = L(a) - L(b)$,
4. $L(a^b) = bL(a)$,
5. $L(x) < 0$, for $x < 1$,
6. $L(x) > 0$, for $x > 1$,
7. If $x < y$, then $L(x) < L(y)$
8. $\frac{d}{dx} L(x) = \frac{1}{x}$

Let e be the real number so that $L(e) = 1$. Define a new function $E(y)$ by $x = E(y)$ if and only if $y = L(x)$. We have:

1. $E(0) = 1, E(1) = e$,
2. $E(a + b) = E(a) \times E(b)$,
3. $E(a - b) = E(a)/E(b)$,
4. $E(ab) = E(a)^b$,
5. $E(x) < 1$, for $x < 0$,
6. $E(x) > 1$, for $x > 0$,
7. If $x < y$, then $E(x) < E(y)$
8. To find the derivative, we have:

$$\begin{aligned}
 1 &= \frac{dx}{dx} \\
 &= \frac{d}{dx} L(E(x)) \\
 &= \frac{1}{E(x)} \frac{d}{dx} E(x) \\
 \frac{d}{dx} E(x) &= E(x)
 \end{aligned}$$

Note, that since $E(1)=e$, we then can show that $E(x)=e^x$. Then from our standard definition, this would make $L(x) = \log_e(x) = \ln(x)$.

Using the Logarithm

Prior to the advent of silicon based calculators, carbon-based calculators did all of the hard work. They used logarithms to make the calculations easier.

We used to have to work with tables of values for logarithms and trigonometric functions. These were used for various types of problems that we will consider later. First we need to see what log tables look like.

We will consider logarithms base 10. This is because we normally think about numbers in terms of base 10, not in terms of base e . What do we know?

$$\log(1) = 0.$$

$$\log(10) = 1.$$

$$\log(20) = \log(2 * 10) = \log(2) + \log(10) = 1 + \log(2),$$

so I need to know $\log(2)$.

$$\log(22) = \log(2.2 * 10) = 1 + \log(2.2),$$

so I need to know $\log(2.2)$.

In fact, if we know the logs of $a.b$ where a is a digit greater than or equal to one and b is a digit different from 0.

How do we find it and once we know how and how precise our answer is?

$$\log(1.1) = 0.04139$$

$$\log(1.2) = 0.07918$$

$$\log(1.3) = 0.11394$$

$$\log(1.4) = 0.14613$$

$$\log(1.5) = 0.17609$$

etc.

From this we get that $\log(11)=1.04139$, $\log(110)=2.04139$. What is $\log(111)$? It is between $\log(110)$ and $\log(120)$, because the log is one-to-one and increasing. We may assume that the function is locally linear, that is that over a small interval the function is essentially linear.

In our case we need the equation of the line between $(110, 2.04139)$ and $(120, 2.07918)$.

$$m = \frac{2.07918 - 2.04139}{120 - 110} = 0.003779$$

The equation is then $y = 0.003779(x - 110) + 2.04139$. Thus, $\log(111) = 2.04139 + 0.003779 = 2.04517$. This is known as *linear interpolation*.

Let's look at the product of two numbers.

$$A = 2.342 \times 24.57$$

If $x = \log(a)$, then $[x] = \text{characteristic}$ of the number and $x - [x]$ is called the *mantissa*. When using tables to compute numbers using logarithms, the reverse process to finding the logarithm is called the antilog. Yes, this is really the exponential function, 10^x . However, we don't exponentiate, we go backwards on the process of looking up the logarithm, but we never really discuss the process of taking 10 to a power.

$$\begin{aligned}\log(A) &= \log(2.342) + \log(24.57) = 0. \\ &= 0.36948689 + 1.39040516 \\ &= 1.75989205\end{aligned}$$

$$A = \text{antilog}(1.75989205)$$

Now, $\text{antilog}(1.75981888) = 57.52$ and $\text{antilog}(1.75989437) = 57.53$. We must interpolate to find A .

$$\frac{A - 57.72}{1.75989205 - 1.75981888} = \frac{57.73 - 57.72}{1.75989437 - 1.75981888}$$

$$A = 57.72 + 0.009693 = 57.7297$$

Example 2:

$$B = \frac{23.44 \times 34.92}{734.5 \times 9.221}$$

$$\begin{aligned}\log(B) &= \log(23.44) + \log(34.92) - (\log(734.5) + \log(9.221)) \\ &= 1.36995761 + 1.54307424 - (2.86599180 + 0.96477802) \\ &= 2.91303185 - 3.83076982 = -0.91773797 \\ &= -1 + 0.08226203\end{aligned}$$

$$\text{antilog}(0.08226203) = 1.208 + \left(\frac{0.01}{0.08242630 - 0.08206693} \right) (0.08242630 - 0.08226203) = 1.20854$$

So, $B = 0.120854$

Computing logarithms

Normally, we would assume that we would use a Taylor series to compute these logarithms. The Taylor series for the logarithm is

$$\ln(x) = \sum_{n=1}^{\infty} -\frac{(1-x)^n}{n}$$

To calculate the numerical value of the natural logarithm of a number, the Taylor series expansion can be rewritten as:

$$\ln(1+x) = x \left(\frac{1}{1} - x \left(\frac{1}{2} - x \left(\frac{1}{3} - x \left(\frac{1}{4} - x \left(\frac{1}{5} - \dots \right) \right) \right) \right) \right) \text{ for } |x| < 1.$$

To obtain a better rate of convergence, the following identity can be used

$$\begin{aligned}\ln(x) &= \ln\left(\frac{1+z}{1-z}\right) = 2z\left(\frac{1}{1} + \frac{1}{3}z^2 + \frac{1}{5}z^4 + \frac{1}{7}z^6 + \frac{1}{9}z^8 + \dots\right) \\ &= 2z\left(\frac{1}{1} + z^2\left(\frac{1}{3} + z^2\left(\frac{1}{5} + z^2\left(\frac{1}{7} + z^2\left(\frac{1}{9} + \dots\right)\right)\right)\right)\right)\end{aligned}$$

provided that $z = (x-1)/(x+1)$ and $x > 0$.

Let's check it out. Let's find $\ln(1.01)$. From this we can find $\log(1.01)$.

If $x = 1.01$, then $z = (1.01 - 1)/(1.01+1) = 0.01/2.01 = 0.004975$.

First iteration: $\ln(1.01) \approx 2z(1) = 0.009950248756219$.

Second iteration: $\ln(1.01) \approx 2z\left(1 + \frac{z^2}{3}\right) = 0.0099503308519493$. The TI-*n*spire gives us $\ln(1.01) = 0.0099503308531681$.

Trigonometric Functions

The trigonometric functions have a long history in mathematics. The study of trigonometric functions dates back to Babylonian times, and a considerable amount of fundamental work was done by Persian and Greek mathematicians. In modern mathematics, they are functions of an angle; they are important when studying triangles and modeling periodic phenomena.

In Geometry we will commonly define them as ratios of two sides of a right triangle containing the angle, and later we equivalently define them as the lengths of various line segments from a unit circle. Mathematics has found many equivalent ways of defining them such as expressing them as infinite series or as solutions of certain differential equations, allowing their extension to positive and negative values and even to complex numbers.

In modern usage, there are six basic trigonometric functions. Especially in the case of the last four, these relations are often taken as the definitions of those functions, but one can define them equally well geometrically or by other means and then derive these relations. A few other functions were common historically (and appeared in the earliest tables), but are now seldom used, such as the versine ($1 - \cos \theta$) and the exsecant ($\sec \theta - 1$).

Function	Abbreviation	Relationships
Sine	sin	$\sin \theta = \cos\left(\frac{\pi}{2} - \theta\right)$
Cosine	cos	$\cos \theta = \sin\left(\frac{\pi}{2} - \theta\right)$
Tangent	tan	$\tan \theta = \frac{1}{\cot \theta} = \frac{\sin \theta}{\cos \theta} = \cot\left(\frac{\pi}{2} - \theta\right)$
Cotangent	cot	$\cot \theta = \frac{1}{\tan \theta} = \frac{\cos \theta}{\sin \theta} = \tan\left(\frac{\pi}{2} - \theta\right)$

Secant	sec	$\sec \theta = \frac{1}{\cos \theta} = \csc \left(\frac{\pi}{2} - \theta \right)$
Cosecant	csc	$\csc \theta = \frac{1}{\sin \theta} = \sec \left(\frac{\pi}{2} - \theta \right)$

History

The earliest use of sine appears in the *Sulba Sutras* written in ancient India from the 8th century BCE to the 6th century BCE. Trigonometric functions were later studied by Hipparchus of Nicaea (180-125 BCE), Ptolemy, Aryabhata (476–550), Varahamihira, Brahmagupta, Muhammad ibnMusa al-Kwarizmi, Abu'l-Wafa, Omar Khayyam, Bhaskara II, Nasir al-Din Tusi, Ghiyath al-Kashi (14th century), Ulugh Beg (14th century), Regiomontanus (1464), Rheticus, and Rheticus' student Valentin Otho.

Madhava (c. 1400) made early strides in the analysis of trigonometric functions in terms of infinite series. Leonhard Euler's *Introductio in analysin infinitorum* in 1748 was mostly responsible for establishing the analytic treatment of trigonometric functions in Europe, also defining them as infinite series and presenting Euler's formula, as well as the near-modern abbreviations sin., cos., tang., cot., sec., and cosec.

The use of trigonometric functions arises from the early connection between mathematics and astronomy. Early work with spherical triangles was as important as plane triangles.

The first work on trigonometric functions related to chords of a circle. Given a circle of fixed radius, 60 units were often used in early calculations, then the problem was to find the length of the chord subtended by a given angle. For a circle of unit radius the length of the chord subtended by the angle x was $2 \sin(x/2)$. The first known table of chords was produced by the Greek mathematician Hipparchus in about 140 BCE. Although these tables have not survived, it is claimed that twelve books of tables of chords were written by Hipparchus. This makes Hipparchus the founder of trigonometry in Europe.

The next Greek mathematician to produce a table of chords was Menelaus in about 100 AD. Menelaus worked in Rome producing six books of tables of chords which have been lost but his work on spherics has survived and is the earliest known work on spherical trigonometry. Menelaus proved a property of plane triangles and the corresponding spherical triangle property known the *regula sex quantitatum*.

Ptolemy was the next author of a book of chords, showing the same Babylonian influence as Hipparchus, dividing the circle into 360 and the diameter into 120 parts. The suggestion here is that he was following earlier practice when the approximation 3 for π was used. Ptolemy, together with the earlier writers, used a form of the relation $\sin^2 x + \cos^2 x = 1$, although of course they did not actually use sines and cosines but chords.

Similarly, in terms of chords rather than sine and cosine, Ptolemy knew the formulas

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Ptolemy calculated chords by first inscribing regular polygons of 3, 4, 5, 6 and 10 sides in a circle. This allowed him to calculate the chord subtended by angles of 36° , 72° , 60° , 90° and 120° . He then found a method of finding the chord subtended by half the arc of a known chord and this, together with interpolation allowed him to calculate chords with a good degree of accuracy. Using these methods Ptolemy found that $\sin 30'$ ($30' = \text{half of } 1^\circ$) which is the chord of 1° was, as a number in base 60, $0\ 31' 25''$. Converted to decimals this is 0.0087268 which is correct to 6 decimal places, the answer to 7 decimal places being 0.0087265.

The first actual appearance of the sine of an angle appears in the work of the Hindus. Aryabhata, in about 500, gave tables of half chords which now really are sine tables and used *jya* for our \sin . This same table was reproduced in the work of Brahmagupta (in 628) and detailed method for constructing a table of sines for any angle were give by Bhaskara in 1150.

The Arabs worked with sines and cosines and by 980 Abu'l-Wafa knew that

$$\sin 2x = 2 \sin x \cos x$$

although it could have easily have been deduced from Ptolemy's formula

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

with $x = y$.

The Hindu word *jya* for the sine was adopted by the Arabs who called the sine *jiba*, a meaningless word with the same sound as *jya*. Now *jiba* became *jaib* in later Arab writings and this word does have a meaning, namely a 'fold'. When European authors translated the Arabic mathematical works into Latin they translated *jaib* into the word *sinus* meaning 'fold' in Latin. In particular Fibonacci's use of the term *sinus rectus arcus* soon encouraged the universal use of sine.

Chapters of Copernicus's book giving all the trigonometry relevant to astronomy was published in 1542 by Rheticus. Rheticus also produced substantial tables of sines and cosines which were published after his death. In 1533 Regiomontanus's work *De triangulis omnimodis* was published. This contained work on planar and spherical trigonometry originally done much earlier in about 1464. The book is particularly strong on the sine and its inverse.

The term sine certainly was not accepted straight away as the standard notation by all authors. In times when mathematical notation was in itself a new idea many used their own notation. Edmund Gunter was the first to use the abbreviation *sin* in 1624 in a drawing. The first use of *sin* in a book was in 1634 by the French mathematician Hérigone while Cavalieri used Si and Oughtred S.

It is perhaps surprising that the second most important trigonometrical function during the period we have discussed was the versed sine, a function now hardly used at all. The *versine* is related to the sine by the formula

$$\text{versin } x = 1 - \cos x.$$

It is just the sine turned (versed) through 90° .

The cosine follows a similar course of development in notation as the sine. Viète used the term *sinus residuae* for the cosine, Gunter (1620) suggested *co-sinus*. The notation *Si.2* was used by Cavalieri, *s co arc* by Oughtred and *S* by Wallis. Viète knew formulas for $\sin(nx)$ in terms of $\sin x$ and $\cos x$. He gave explicitly the formulas (due to Pitiscus)

$$\sin 3x = 3 \cos 2x \sin x - \sin 3x$$

$$\cos 3x = \cos 3x - 3 \sin 2x \cos x.$$

The tangent and cotangent came via a different route from the chord approach of the sine. These developed together and were not at first associated with angles. They became important for calculating heights from the length of the shadow that the object cast. The length of shadows was also of importance in the sundial. Thales used the lengths of shadows to calculate the heights of pyramids.

The first known tables of shadows were produced by the Arabs around 860 and used two measures translated into Latin as *umbra recta* and *umbra versa*. Viète used the terms *amsinus* and *prosinus*. The name tangent was first used by Thomas Fincke in 1583. The term *cotangens* was first used by Edmund Gunter in 1620.

Abbreviations for the tangent and cotangent followed a similar development to those of the sine and cosine. Cavalieri used *Ta* and *Ta.2*, Oughtred used *t arc* and *t co arc* while Wallis used *T* and *t*. The common abbreviation used today was used by Albert Girard in 1626, but *tan* was written over the angle. The abbreviation *cot* was first used by Jonas Moore in 1674.

The secant and cosecant were not used by the early astronomers or surveyors. These came into their own when navigators around the 15th Century started to prepare tables. Copernicus knew of the secant which he called the *hypotenusa*. Viète knew the results

$$\frac{\csc x}{\sec x} = \cot x = \frac{1}{\tan x}$$

$$\frac{1}{\csc x} = \frac{\cos x}{\cot x} = \sin x$$

The abbreviations used by various authors were similar to the trigonometric functions already discussed. Cavalieri used *Se* and *Se.2*, Oughtred used *se arc* and *sec co arc* while Wallis used *s* and σ . Albert Girard used *sec*, written above the angle as he did for the tangent.

The term trigonometry first appears as the title of a book *Trigonometria* by Pitiscus, published in 1595. Pitiscus also discovered the formulas for $\sin 2x$, $\sin 3x$, $\cos 2x$, and $\cos 3x$.

The 18th Century saw trigonometric functions of a complex variable being studied. Johann Bernoulli found the relation between $\sin^{-1}z$ and $\log z$ in 1702 while Cotes, in a work published in 1722 after his death, showed that

$$ix = \log(\cos x + i \sin x).$$

de Moivre published his famous theorem

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx$$

in 1722 while Euler, in 1748, gave the formula (equivalent to that of Cotes)

$$e^{ix} = \cos x + i \sin x.$$

The hyperbolic trigonometric functions were introduced by Lambert.

Once we know something about similarity in triangles we should see some standard correspondence between the length of the sides of a triangle and the angles of the triangle. It is just these ratios that the trigonometric functions express, geometrically.

Just as we defined the exponential function above in terms of certain functional equations, we can define the trigonometric functions using functional equations based on properties like the sum and difference formulas. If we take these formulas and the Pythagorean identity as given then we can prove that only two real functions satisfy those conditions. Symbolically, we say that there exists exactly one pair of real functions s and c such that for all real numbers x and y , the following equations hold:

$$\begin{aligned} [s(x)]^2 + [c(x)]^2 &= 1 \\ s(x+y) &= s(x)c(y) + c(x)s(y) \\ c(x+y) &= c(x)c(y) - s(x)s(y) \end{aligned}$$

with the added condition that

$$0 < xc(x) < s(x) < x \text{ for } 0 < x < 1.$$

Other derivations, starting from other functional equations, are also possible, and such derivations can be extended to the complex numbers.

Properties of Trigonometric Functions

The *Law of Sines* for an arbitrary triangle states

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Now, it is also known in the formulation:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

The common number, $a/\sin A$, occurring in the theorem is the diameter of the circumcircle through the three points A , B and C . The Law of Sines is useful for computing the lengths of the unknown sides in a triangle if two angles and one side are known. It is the trigonometric representation of the Angle-Side-Angle Theorem in Euclidean geometry. This is a common situation occurring in triangulation, a technique to determine unknown distances by measuring two angles and an accessible enclosed distance.

The *Law of Cosines* is merely an extension of the Pythagorean theorem

$$c^2 = a^2 + b^2 - 2ab \cos C,$$

or

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

The Law of Cosines is useful to determine the unknown data of a triangle if two sides and an angle are known. If the angle is not contained between the two sides, the triangle may not be unique. Note also, that the Law of Cosines measures how far the angle C is from being a right angle. It is also the trigonometric representation of the Side-Angle-Side Theorem in geometry – if you know two sides and the included angle of a triangle, then you know the triangle up to congruence.

There is also a *Law of Tangents* though it is rarely used.

$$\frac{a+b}{a-b} = \frac{\tan\left[\frac{1}{2}(A+B)\right]}{\tan\left[\frac{1}{2}(A-B)\right]}$$

Geometric Realizations of the Trigonometric Functions

For the functions of the unit circle, note that each of the six functions of the angle correspond to certain segments associated with that angle in the unit circle.

If $\angle BOA = \theta$, then $OA = OB = 1$, $OD = EB = \cos \theta$ (the x -coordinate of the point B) and $OE = BD = \sin \theta$ (the y -coordinate of the point B). Now, AC lies on the tangent to the circle at A and is, hence, perpendicular to OA . Also, $\triangle BOD \sim \triangle COA$, so

$$AC = \frac{AC}{OA} = \frac{BD}{OD} = \frac{\sin \theta}{\cos \theta} = \tan \theta$$

Also, from this similarity we have that

$$OC = \frac{OC}{OA} = \frac{OB}{OD} = \frac{1}{\cos \theta} = \sec \theta$$

Likewise, $\triangle GFO \sim \triangle BEO$, and

$$FG = \frac{FG}{OF} = \frac{BE}{OE} = \frac{\cos \theta}{\sin \theta} = \cot \theta,$$

and

$$OG = \frac{OG}{OF} = \frac{OB}{EB} = \frac{1}{\sin \theta} = \csc \theta$$

Thus, we have the following geometric realizations of the six functions:

Function	Segment
$\sin \theta$	$BD = OE$
$\cos \theta$	$OD = BE$
$\tan \theta$	AC

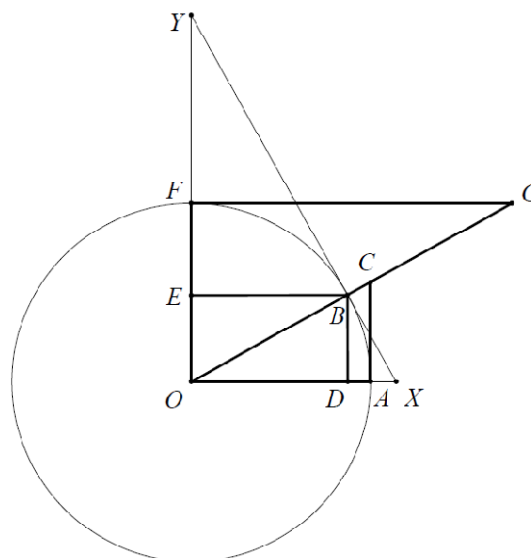


Figure 2: Unit circle realizations of trigonometric values

$\cot \theta$	GF
$\sec \theta$	OC
$\csc \theta$	OG

Note that if we look at the tangent to the circle at B , then we have the point X where the tangent intersects OA and the point Y where it intersects OF . The triangle $\triangle XBO \sim \triangle BDO$, so from this we get that $XB = \tan \theta$ and $OX = \sec \theta$. Likewise $\triangle OBY \sim \triangle BDO$ and this gives us that $BY = \cot \theta$ and $OY = \csc \theta$.