Hyperbolic Trigonometric Functions

Let us quickly recall the analytic definitions of the trigonometric functions of a real variable – that is the definition that did not use the right triangles.

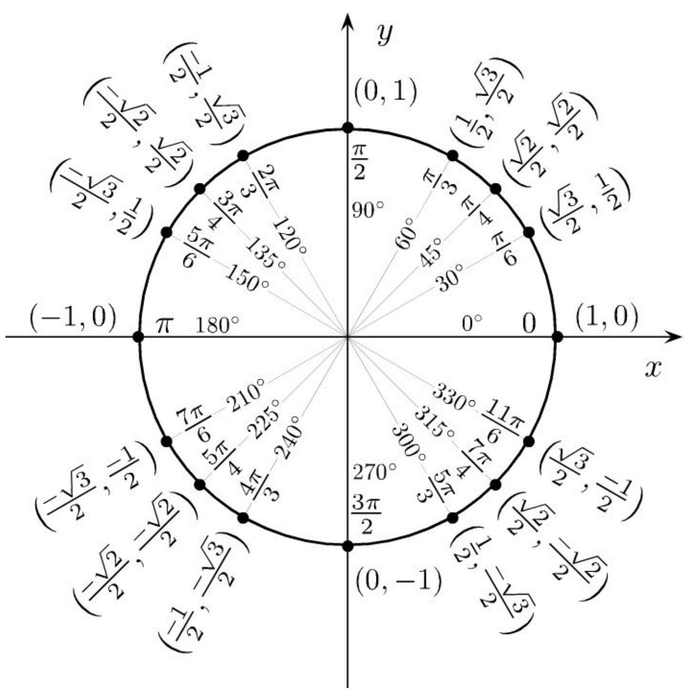
You have seen the definition of the six trigonometric functions in terms of a right triangle:

$$\sin(A) = \frac{opposite}{hypotenuse} \qquad \cos(A) = \frac{adjacent}{hypotenuse} \qquad \tan(A) = \frac{opposite}{adjacent}$$

We can also define these functions in terms of the unit circle, the circle of radius one centered at the origin. This does not necessarily help us calculate the sine and cosine of an angle, because it takes us no further than the right triangle definitions; indeed it relies on right triangles for most angles. The unit circle definition does, however, allow us to define the trigonometric functions for all positive and negative real numbers, not just for values between 0 and π/2 radians. From the Pythagorean theorem the equation for the unit circle is:

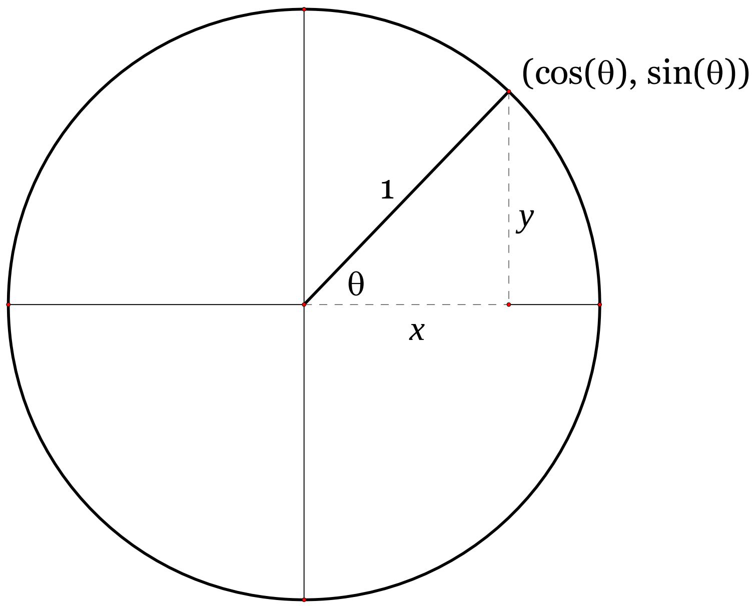
$$x^2 + y^2 = 1$$

In the figure below, some common angles, measured in radians, are given. Measurements in the counter clockwise direction are positive angles and measurements in the clockwise direction are negative angles.



Let a line through the origin, making an angle of θ with the positive half of the *x*-axis intersect the unit circle. Then the point of intersection of this line with the unit circle is $(\cos\theta, \sin\theta)$. In the triangle below, the radius is equal to the hypotenuse and has length

1, so we have $\sin\theta = \frac{y}{1}$ and $\cos\theta = \frac{x}{1}$. The unit circle can be thought of as a way of looking at an infinite number of triangles by varying the lengths of their legs but keeping the lengths of their hypotenuses equal to 1.



Now, I want us to think of this just a little differently. Instead of the angle between the x-axis and the segment, what is the area of that sector of the circle? This means that we have to remember the area of a sector of a circle. If an angle, θ, measured in radians, sweeps out a sector in the circle of radius r, the area of the sector is

$$A = \frac{1}{2}\theta r^2$$

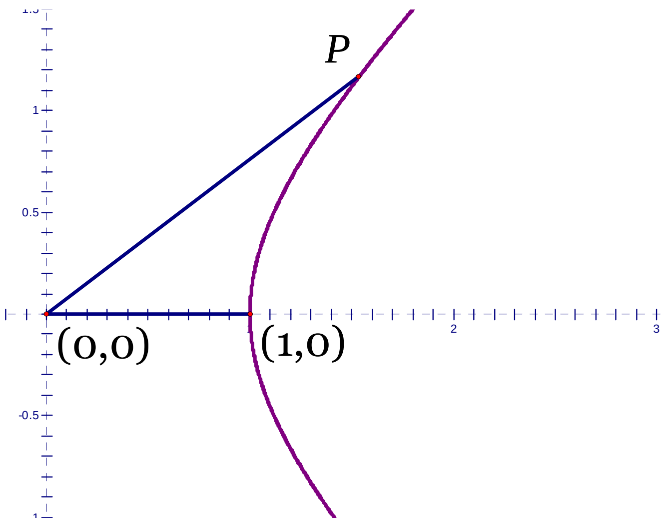
So in the unit circle, since $r = 1$, the area is $A = \theta/2$, or $\theta = 2A$. Thus, the coordinates of the point on the unit circle swept out by an area of A radians are $(\cos(2A), \sin(2A))$.

Now, instead of a circle of radius 1, think about a different conic cross section – the hyperbola. We are interested in the "unit" hyperbola, the hyperbola given by the equation,

$$x^2 - y^2 = 1.$$

Note that there is very little different in the equation of the hyperbola and the circle, just subtraction instead of addition. Can we do a similar construction for the hyperbola that we did for the circle?

First, we need to recall that the asymptotes for the hyperbola are the lines $y = x$ and $y = -x$. If a line passes through the origin with slope greater than 1 or less than –1, that line will not intersect the hyperbola. In the figure below, consider a ray emanating at the origin and having slope m, with $-1 < m < 1$. This ray will intersect the hyperbola at a point P. Consider the area of the region bounded by the ray, the hyperbola and the



positive x-axis and call this area A. Then the coordinates of the point P are $(\cosh(2A), \sinh(2A))$.

Again, this does not help us compute the values of the hyperbolic sine and cosine. To do this, we appeal to some mathematics that we might cover later. Mathematicians have found that there is a relationship between the exponential function and the trigonometric functions. This then leads to considering the following two functions:

$$s(x) = \frac{e^x - e^{-x}}{2}$$

$$c(x) = \frac{e^x + e^{-x}}{2}$$

What do the graphs of these functions look like? We could plot them on the calculator, but because of their pattern, we want to try some algebra first.

Note that

$$(s(x))^2 = \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{e^{2x} - 2 + e^{-2x}}{4}$$

and

$$(c(x))^2 = \left(\frac{e^x + e^{-x}}{2}\right)^2 = \frac{e^{2x} + 2 + e^{-2x}}{4},$$

so that

$$(c(x))^2 - (s(x))^2 = \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = \frac{2}{4} + \frac{2}{4} = 1$$

That is, we get that these functions satisfy the equation

$$u^2 - v^2 = 1.$$

In fact, each point on the hyperbola has coordinates $(c(x), s(x))$ for some value of x. Not surprisingly, these functions are the hyperbolic cosine ($c(x)$) and the hyperbolic sine ($s(x)$). Thus, we often define these functions by

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

We also have seen the basic hyperbolic trigonometric identity:

$$\cosh^2(x) - \sinh^2(x) = 1$$

Once we have the hyperbolic sine and hyperbolic cosine defined, then we define the other four functions as:

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{e^x + e^{-x}}{e^x - e^{-x}}, \ x \neq 0$$

$$\text{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^x + e^{-x}}$$

$$\text{csch}(x) = \frac{1}{\sinh(x)} = \frac{2}{e^x - e^{-x}}, \ x \neq 0$$

Based on these definitions and the basic hyperbolic trigonometric identity, we find a large number of hyperbolic trigonometric identities that are analogous to the usual trigonometric identities

Trigonometric Identity	Hyperbolic Trigonometric Identity
$\cos^2 x + \sin^2 x = 1$	$\cosh^2 x - \sinh^2 x = 1$
$1 + \tan^2 x = \sec^2 x$	$1 - \tanh^2 x = \text{sech}^2 x$
$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$	$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$	$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$
$\tan(x + y) = \frac{\tan x + \tan y}{1 + \tan x \tan y}$	$\tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$
$\sin^2 \frac{x}{2} = \frac{1 - \cos x}{2}$	$\sinh^2 \frac{x}{2} = \frac{\cosh x - 1}{2}$
$\cos^2 \frac{x}{2} = \frac{1 + \cos x}{2}$	$\cosh^2 \frac{x}{2} = \frac{\cosh x + 1}{2}$
$\tan^2 \frac{x}{2} = \frac{1 - \cos x}{1 + \cos x}$	$\tanh^2 \frac{x}{2} = \frac{\cosh x - 1}{\cosh x + 1}$
$\tan \frac{x}{2} = \frac{\sin x}{1 + \cos x}$	$\tanh \frac{x}{2} = \frac{\sinh x}{\cosh x + 1}$
$\tan \frac{x}{2} = \frac{1 - \cos x}{\sin x}$	$\tanh \frac{x}{2} = \frac{\cosh x - 1}{\sinh x}$
$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}$	$\sinh x = 2 \sinh \frac{x}{2} \cosh \frac{x}{2}$
$\sin x \pm \sin y = 2 \sin \frac{1}{2}(x \pm y) \cos \frac{1}{2}(x \mp y)$	$\sinh x \pm \sinh y = 2 \sinh \frac{1}{2}(x \pm y) \cosh \frac{1}{2}(x \mp y)$
$\cos x + \cos y = 2 \cos \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y)$	$\cosh x + \cosh y = 2 \cosh \frac{1}{2}(x + y) \cosh \frac{1}{2}(x - y)$
$\cos x - \cos y = -2 \sin \frac{1}{2}(x + y) \sin \frac{1}{2}(x - y)$	$\cosh x - \cosh y = 2 \sinh \frac{1}{2}(x + y) \sinh \frac{1}{2}(x - y)$

What other properties do they seem to have in common? We should consider at least three more things: inverse functions and derivatives and graphs.

Inverse Hyperbolic Trigonometric Functions

Since the hyperbolic trigonometric functions are defined in terms of exponentials, we might expect that the inverse hyperbolic functions might involve logarithms. Let us first consider the inverse function to the hyperbolic sine: arcsinh(*x*).

By the definition of an inverse function, $y = \operatorname{arcsinh}(x)$ means that $x = \sinh(y)$. Thus,

$$x = \frac{e^y - e^{-y}}{2}$$
$$e^y - e^{-y} = 2x$$
$$\left(e^y - e^{-y}\right)e^y = 2xe^y$$
$$e^{2y} - 2xe^y - 1 = 0$$

Let $u = e^y$, then this equation becomes

$$u^2 - 2xu - 1 = 0$$
$$u = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}$$
$$e^y = x + \sqrt{x^2 + 1}$$

$y = \ln(x + \sqrt{x^2 + 1})$, and it must be the positive square root because $e^y > 0$.

$$\operatorname{arcsinh}(x) = \ln(x + \sqrt{x^2 + 1})$$

We should not find this too surprising. We would expect the others to be similar. Doing similar work, we find that:

$$\operatorname{arccosh}(x) = \ln(x + \sqrt{x^2 - 1})$$
$$\operatorname{arctanh}(x) = \ln\left(\sqrt{\frac{1+x}{1-x}}\right) = \frac{1}{2}\ln(1+x) - \frac{1}{2}\ln(1-x)$$
$$\operatorname{arccoth}(x) = \ln\left(\sqrt{\frac{x+1}{x-1}}\right) = \frac{1}{2}\ln(x+1) - \frac{1}{2}\ln(x-1)$$
$$\operatorname{arcsech}(x) = \ln\left(\frac{1 + \sqrt{1 - x^2}}{x}\right)$$
$$\operatorname{arccsch}(x) = \ln\left(\frac{1 + \sqrt{1 + x^2}}{x}\right)$$

Wow! No, you don't have to memorize all of this!!

Derivatives of Hyperbolic Trigonometric Functions

Just like everything else, we would expect the derivatives of the hyperbolic trigonometric functions to take an analogous route to those of the regular trigonometric functions.

We can easily find the derivatives since they are defined in terms of the exponential function.

$$\frac{d}{dx}\sinh(x)=\frac{d}{dx}\frac{e^x-e^{-x}}{2}=\frac{1}{2}\frac{d}{dx}(e^x-e^{-x})=\frac{1}{2}(e^x-(-e^{-x}))=\frac{1}{2}(e^x+e^{-x})=\cosh(x)$$
$$\frac{d}{dx}\cosh(x)=\frac{d}{dx}\frac{e^x+e^{-x}}{2}=\frac{1}{2}\frac{d}{dx}(e^x+e^{-x})=\frac{1}{2}(e^x+(-e^{-x}))=\frac{1}{2}(e^x-e^{-x})=\sinh(x)$$

Thus we have found that these functions have a nice derivative periodicity – and we do not have to take negative signs into account:

$$\frac{d}{dx}\sinh(x)=\cosh(x)$$
$$\frac{d}{dx}\cosh(x)=\sinh(x)$$

From these, we can compute the other derivatives – expecting analogous results.

$$\frac{d}{dx}\tanh(x)=\frac{d}{dx}\frac{\sinh(x)}{\cosh(x)}=\frac{\cosh x\cosh x-\sinh x\sinh x}{(\cosh x)^2}=\frac{1}{(\cosh x)^2}=\operatorname{sech}^2(x)$$
$$\frac{d}{dx}\coth(x)=\frac{d}{dx}\frac{\cosh(x)}{\sinh(x)}=\frac{\sinh x\sinh x-\cosh x\cosh x}{(\sinh x)^2}=\frac{-1}{(\sinh x)^2}=-\operatorname{csch}^2(x)$$
$$\frac{d}{dx}\operatorname{sech}(x)=\frac{d}{dx}\frac{1}{\cosh(x)}=\frac{-\sinh x}{(\cosh x)^2}=-\operatorname{sech}(x)\tanh(x)$$
$$\frac{d}{dx}\operatorname{csch}(x)=\frac{d}{dx}\frac{1}{\sinh(x)}=\frac{-\cosh x}{(\sinh x)^2}=-\operatorname{csch}(x)\coth(x)$$

These are pretty close to what we expected.

Graphs of Hyperbolic Trigonometric Functions

We wait until now to look at the graphs so that we can use the derivative to help us. First, let us look at cosh(x).

$\cosh(x)=\dfrac{e^x+e^{-x}}{2}>0$ since the numerator is always positive.

Since $\frac{d}{dx}\sinh(x)=\cosh(x)$, and $\cosh(x)>0$ for all x, we see that the hyperbolic sine function is always increasing. We also note that it has no critical points, since its derivative is always defined and is never 0. Now, looking at the graph it is not too surprising to find that it looks like the figure to the right.

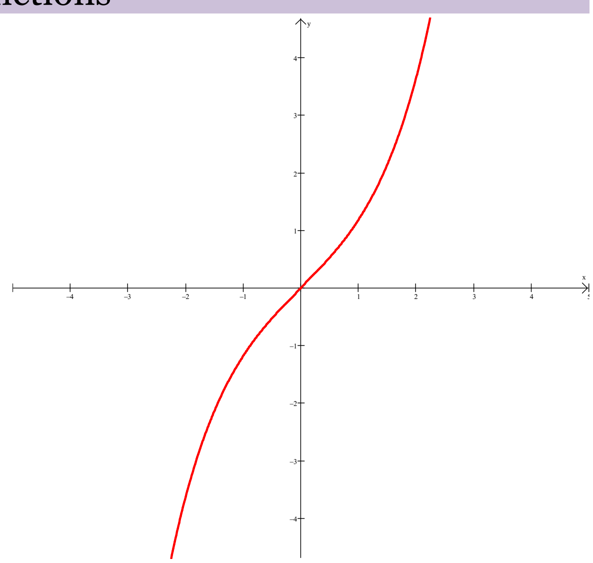


Figure 1: $y=\sinh(x)$

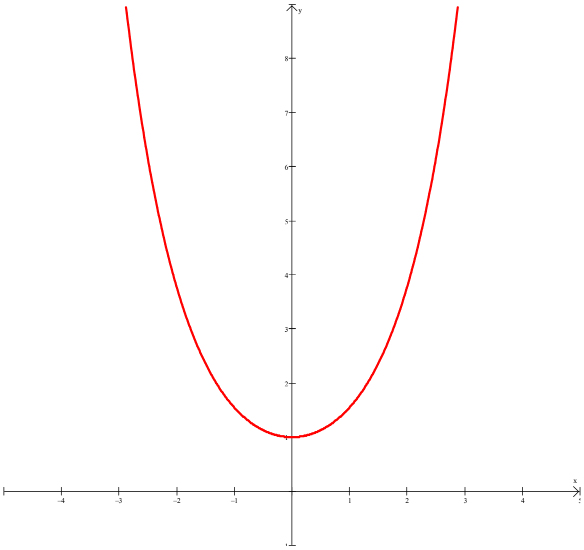
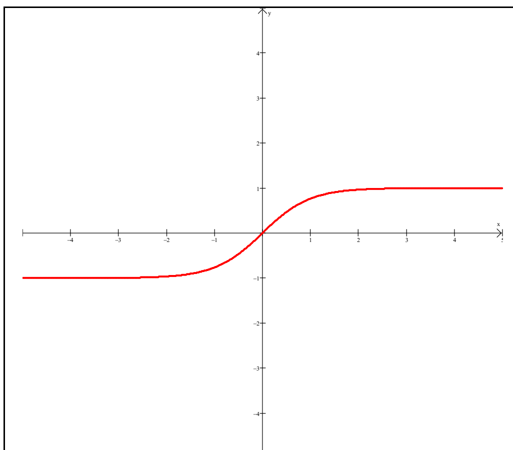


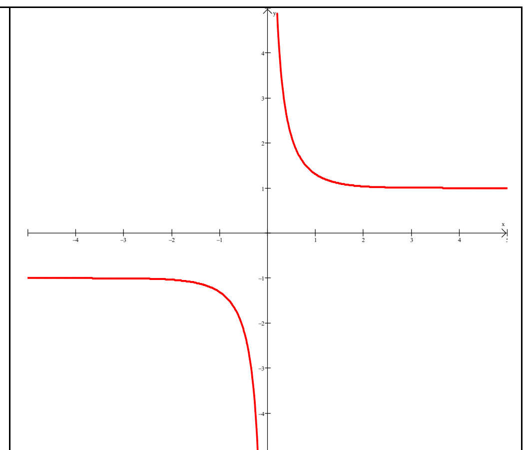
Figure 2: $y = \cosh(x)$

Now that we know what the hyperbolic sine looks like, we can analyze the hyperbolic cosine. Since its derivative is 0 at $x = 0$, we know that it has a critical point there. Since the second derivative is always positive this critical point must be a local minimum. It is not hard to show that it is a global minimum. The graph looks like the figure on the left.

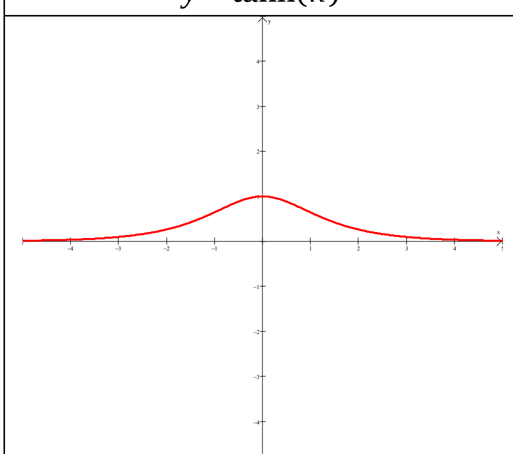
This may look somewhat like a parabola to you, but it actually grows faster than any parabola. The interesting thing is that it does describe a physical setting. If you put two pegs at the same height some distance apart and let a rope hang between the two pegs so that the ends just hang over the pegs (not tied to them), then this rope takes on a shape called a catenary. The catenary is a hyperbolic cosine shape. Examples are electrical wires hanging between power poles.



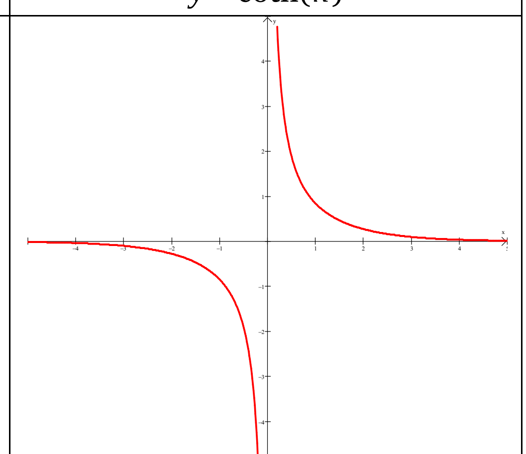
$y = \tanh(x)$



$y = \coth(x)$



$y = \text{sech}(x)$



$y = \text{csch}(x)$

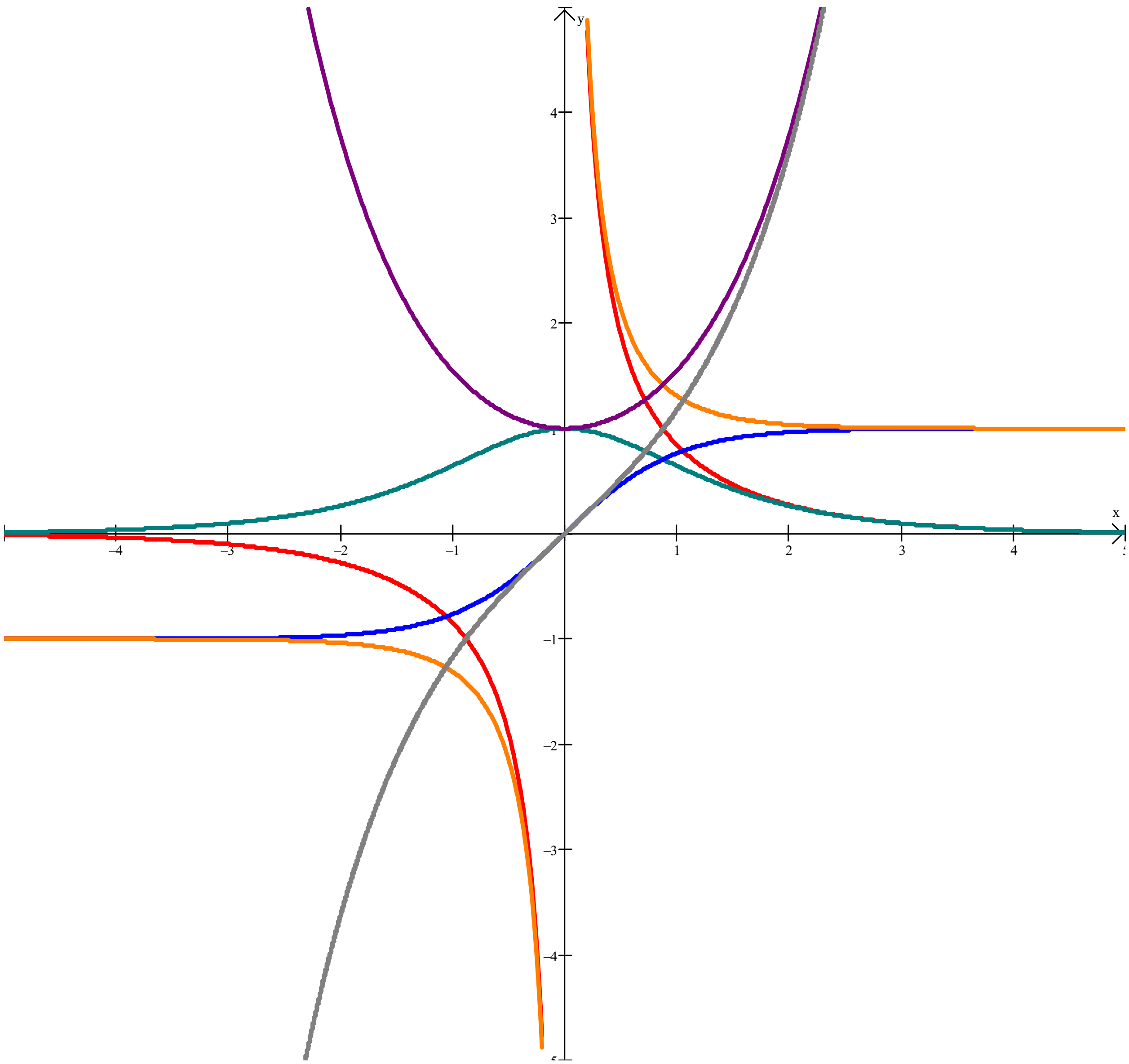


Figure 3: A Web of Hyperbolic Trigonometric Functions