MATH 6102
Spring 2009

Functions, Sequences and Limits

Certain Subsets of the Reals

We will make some simple definitions. Let a and b be any two real numbers with $a < b$.

$$(a,b) = \{ x \in R \mid a < x < b\}$$

$$[a,b] = \{ x \in R \mid a \leq x \leq b\}$$

$$(a,b] = \{ x \in R \mid a < x \leq b\}$$

$$[a,b) = \{ x \in R \mid a \leq x < b\}$$

$$(a,\infty) = \{ x \in R \mid a < x \}$$

$$[a,\infty) = \{ x \in R \mid a \leq x \}$$

$$(-\infty,b) = \{ x \in R \mid x < b \}$$

$$(-\infty,b] = \{ x \in R \mid x \leq b \}$$

Subsets of the Reals

If $r \in$ R then a *neighborhood* of r is ***<u>an</u>*** open interval (a,b) so that $r \in (a,b)$.

The neighborhood is *centered* at r if

$$r = (a + b)/2$$

If ε and a are reals, then the *ε-neighborhood* of a is the interval $(a - \varepsilon, a + \varepsilon)$

History of Function

Oresme – 1350

Galileo – 1500's

Descartes – 1600's

Newton – 1660's

Leibniz – 1673 - the first to use the term *function*

Jean Bernoulli – 1718

Leonhard Euler – 1748 & 1755

History of Function

- Cauchy – 1821 – still thinking of a function in terms of a formula (either explicit or implicit)
- Fourier – 1822 – introduced general Fourier series but fell back on old definitions
- Dirichlet – 1837 – defined general function and continuity (in modern terms)
- Weierstrauss – 1885 – any continuous function is the limit of a uniformly convergent sequence of polynomials
- Goursat – 1923 – modern definition

Definitions

Bernoulli – 1718 – *One calls here a function of a variable a quantity composed in any manner whatever of this variable and constants*.

Basically this meant +, –, ×, ÷, √, logs and sines.

They would say that $f(x)$ depended *analytically* on the variable x.

Definitions

Dirichlet – 1837

$$f(x)=\begin{cases}1 & \text{if } x \text{ is rational and } 0\leq x\leq 1\\ 0 & \text{if } x \text{ is irrational and } 0\leq x\leq 1\end{cases}$$

A More Modern Definition

Let D be a set of real numbers. A function

$$f\colon D \rightarrow \mathrm{R}$$

is a rule that assigns a number $f(x)$ to every element x of D.

Sequences

Let N = the set of natural numbers (it will not matter if it starts with 0 or with 1).

A sequence is a function $a: N \rightarrow$ R.

We will normally denote a sequence by its set of outputs $\{a_n\}$, where $a_n = a(n)$.

Occasionally you will see $a_0, a_1, a_2, a_3, \ldots$ or $\{a_n\}^{\infty}_{n=0}$

Examples

1) $\{1,2,3,4,5,6,\ldots\}$ – an arithmetic progression ($f(n) = n$)
2) $\{a + bn \mid n=0,1,2,3,\ldots\}$ – a different type of arithmetic progression – ($f(n) = a + bn$)
3) $\{a^0,a^1,a^2,a^3,a^4,\ldots\}$ – a geometric progression ($f(n) = a^n$)
4) $\{1, 1/2, 1/3, 1/4, 1/5,\ldots\}$ - ($f(n) = 1/n$)
5) $f(n) = a_n = (-1)^n$. Note that the range is $\{-1,1\}$

Examples

1) $f(n) = a_n = \cos(\pi n/3)$
$a_1 = \cos(\pi/3) = \cos 60° = ½$
$\{a_n\}$ = {½, -½, -1, -½, ½, 1,½, -½, -1, -½, ½, 1, …}. The function takes on only a finite number of values, but the sequence has an infinite number of elements.
2) $f(n) = a_n = n^{1/n}$,
{ 1, $2^{1/2}$, $3^{1/3}$,$4^{1/4}$,…} = {1, 1.41421, 1.44225, 1.41421, 1.37973, 1.34801, 1.32047, 1.29684, 1.27652, 1.25893,…}
Also a_{100}=1.04713, $a_{10,000}$ = 1.00092
3) $b_n = (1+1/n)^n$
{2, $(3/2)^2$, $(4/3)^3$, $(54)^4$,…} = {2, 2.25, 2.37037, 2.44141, 2.48832, 2.52163, 2.54650, 2.56578, 2.58117, 2.59374,…}
Also a_{100}=2.74081 and $a_{10,000}$ = 2.71815

Almost all …

Definition: It is said that *almost all the terms* of the sequence $\{a_n\}$ have a certain property provided that there is an index N such that $\{a_n\}$ possesses this property whenever $n \geq N$.

Convergence

Definition 1: A sequence of real numbers is said to *converge* to a real number L if for every $\varepsilon > 0$ there is an integer $N > 0$ such that if $k > N$ then $|a_k - L| < \varepsilon$.

Definition 2: A sequence of real numbers is said to *converge* to a real number L if every neighborhood of L contains almost all of the terms of $\{a_n\}$.

The number L is called the *limit* of the sequence.

Convergence

Lemma 1: The sequence $\{1/n\}$ converges to 0.

Proof : Let (a,b) be any neighborhood of 0. This means that $a < 0 < b$. Let $N>[1/b]$, be an integer greater than $1/b$. Then $1/N < b$ and for every integer $n > N$, we have that

$$a < 0 < 1/n < 1/N < b$$

and (a,b) contains almost all of the elements of the sequence. Thus, the sequence converges to 0.

Convergence

Lemma 1: The sequence $\{1/n\}$ converges to 0.

Proof : You prove this using Definition 1.

Convergence

Definition: A sequence is *convergent* if it has a limit. If it is not convergent it is called *divergent*.

Lemma 2: The sequence $\{a_n\}$ converges to L if and only if every neighborhood of L that is centered at L contains almost all of the terms of the sequence.

Note that this tells us that the two definitions are the same.

Example

Let $a_n = n/2^n$. $\{a_n\} = \{1/2, 2/2^2, 3/2^3, 4/2^4,\ldots\}$
Educated guess: $\{a_n\} \rightarrow 0$.

Let $\varepsilon = 0.1, 0.01, 0.001, 0.0001, 0.00001$.

We need to find an integer N so that

$$| N/2^N - 0 | < \varepsilon$$

Look in the table of values. Note that for $N = 6$ the above is true if $\varepsilon = 0.1$

ε	N
1	N>0
0.1	N>5
0.01	N>9
0.001	N>14
0.0001	N>18
0.00001	N>22

Theorem (Convergent sequences are bounded)

Let $\{a_n\}$ be a convergent sequence. Then the sequence is bounded, and the limit is unique.

Proof:

(i)Uniqueness: Suppose the sequence has two limits, L and K. Let $\varepsilon > 0$. There is an integer N_K such that $|a_n - K| < \varepsilon/2$ if $n > N_K$.

Also, there is an integer N_L such that $|a_n - L| < \varepsilon/2$ if $n > N_L$.

By Triangle Inequality:

$$|L - K| < |a_n - L| + |a_n - K| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

if $n > \max\{N_K, N_L\}$.

Therefore $|L - K| < \varepsilon$ for any $\varepsilon > 0$. But this means that $L = K$.

Convergent sequences are bounded

Proof:

(ii) Boundedness. Since the sequence converges, choose any $\varepsilon > 0$. Specifically take $\varepsilon = 1$. There is N so that

$$| a_n - L | < 1 \text{ if } n > N.$$

Fix N. Then

$$| a_n | \le | a_n - L | + |L| < 1 + |L| = P \text{ for all } n > N.$$

Let $M=\max\{|a_1|,|a_2|,\ldots|a_N|, P\}$. Thus $|a_n| < M$ for all n, which makes the sequence bounded.

Theorem: *If* $\{a_n\} \longrightarrow L$, $\{b_n\} \longrightarrow M$ *and* α *is a real number, then*

1. $lim_{n \longrightarrow \infty}\ \alpha = \alpha.$
2. $lim_{n \longrightarrow \infty}\ (a_n \pm b_n) = L \pm M$
3. $lim_{n \longrightarrow \infty}\ (a_n \times b_n) = L \times M$
4. $lim_{n \longrightarrow \infty}\ (\alpha a_n) = \alpha L$
5. *If* $a_n \leq b_n$ *for all* $n \geq m$, *then* $L \leq M$
6. *If* $b_n \neq 0$ *for all n and if* $M \neq 0$, *then* $glb\{|b_n|\} > 0$.
7. $lim_{n \longrightarrow \infty}\ (a_n / b_n) = L/M$, *provided* $M \neq 0$.

Proof:

1. $lim_{n \longrightarrow \infty} \alpha = \alpha$

   Since $\alpha - \alpha = 0$, for any $\varepsilon > 0$, $|\alpha - \alpha| < \varepsilon$ and we are done.

2. $lim_{n \longrightarrow \infty} (a_n \pm b_n) = L \pm M$

   Do this for the sum. The difference is similar.

   Let $\varepsilon > 0$, there exist N_a and N_b so that

   $$|a_n - L| < \varepsilon/2 \quad \text{if } n > N_a \textit{ and}$$

   $$|b_n - M| < \varepsilon/2 \quad \text{if } n > N_b.$$

   Let $K = \max\{N_a, N_b\}$, then if $n > K$

   $$|(a_n+b_n) - (L+M)| \leq |a_n - L| + |b_n - M| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

3. $lim_{n\longrightarrow\infty}(a_n \times b_n) = L \times M$

Note:

$$|(a_nb_n) - (LM)| \leq |(a_n - L)\, b_n + L(b_n - M)|$$
$$\leq |(a_n - L)\, b_n| + |L(b_n - M)|$$
$$= |(a_n - L)||b_n| + |L||(b_n - M)|$$

Then use the fact that $\{b_n\}$ is bounded.

4. $lim_{n\longrightarrow\infty}(\alpha a_n) = \alpha L$

Consider ε/α if $\alpha \neq 0$. If $\alpha = 0$ this is easy.

5. *If* $a_n \leq b_n$ *for all* $n \geq m$, *then* $L \leq M$

6. *If* $b_n \neq 0$ *for all n and if* $M \neq 0$, *then* $glb\{|b_n|\}>0$.

Let $\varepsilon = |M|/2 > 0$. $\{b_n\} \longrightarrow M$ so there is N so that if $n > N$ then $|b_n - M| < |M|/2$.

So if $n > N$ we must have $|b_n| \geq |M|/2$.

If not by the Triangle Inequality

$$|M| = |M - b_n + b_n| \leq |M - b_n| + |b_n|$$
$$< |M|/2 + |M|/2 = |M|$$

So set

$$m = \min\{|M|/2, |b_1|, |b_2|, \ldots, |b_N|\}.$$

Then $m > 0$ and $|b_n| \geq m$ for all n

7. $lim_{n\longrightarrow\infty}(a_n/b_n) = L/M$, *provided* $M \neq 0$.

Reduce to $lim_{n\longrightarrow\infty}(1/b_n) = 1/M$ – How?

Let $\varepsilon > 0$. By (6) there is $m > 0$ so that $|b_n| \geq m$. Since $\{b_n\}$ is convergent there is N so that if $n > N$

$$|M - b_n| < \varepsilon\, m\, |M|$$

Then for $n > N$

$$|1/b_n - 1/M| = |b_n - M|/|b_n M|$$
$$\leq |b_n - M|/(m|M|) < \varepsilon$$

Example

$$\lim_{n\to\infty}\frac{3n^4+4n^3-7n^2-5280n+3216547}{7n^4+5588741226n^2-7}$$

$$=\lim_{n\to\infty}\frac{3\frac{n^4}{n^4}+4\frac{n^3}{n^4}-7\frac{n^2}{n^4}-5280\frac{n}{n^4}+3216547\frac{1}{n^4}}{7\frac{n^4}{n^4}+5588741226\frac{n^2}{n^4}-7\frac{1}{n^4}}$$

$$=\lim_{n\to\infty}\frac{3+\frac{4}{n}-\frac{7}{n^2}-\frac{5280}{n^3}+\frac{3216547}{n^4}}{7+\frac{5588741226}{n^2}-\frac{7}{n^4}}=\frac{3}{7}$$

The Squeeze Theorem

<u>Theorem</u>: *If* $\{a_n\} \longrightarrow L$, $\{b_n\} \longrightarrow L$ *and*

$$a_n \le c_n \le b_n \text{ for all } n \ge m$$

Then $\{c_n\} \longrightarrow L$.

The Power Theorem

<u>Theorem</u>: *Let a be fixed. Then*

$$\lim_{n\to\infty} a^n = \begin{cases} 0 & \text{if } |a| < 1 \\ 1 & \text{if } a = 1 \\ dne & \text{if } |a| > 1 \\ dne & \text{if } a = -1 \end{cases}$$

Find

$$\lim_{n \to \infty} \frac{3n^2 + 5n + 1}{n + 1}$$

Find

$$\lim_{n \to \infty} \frac{3n^2 + 5n + 1}{n^2 + 1}$$

Find

$$\lim_{n\to\infty} \frac{3n^2+5n+1}{n^3+1}$$

Find

$$\lim_{n\to\infty} \frac{0.5^n + 3\sin(n)}{\sqrt{n}}$$

Find

$$\lim_{n\to\infty}\frac{2^n-1}{3^n+1}$$

Find

$$\lim_{n\to\infty}\frac{2^n+1}{3^n-1}$$

Find

$$\lim_{n\to\infty}\frac{3^n+2^n}{3^n-2^n}$$

Find

$$\lim_{n\to\infty}\frac{3^{n}+4^{n-3}}{5^{n+2}-2^{n+4}}$$

Find

$$\lim_{n\to\infty} \frac{4^{2n-3} + 2^{5n+6}}{5^{3n-2} - 3^{n+10}}$$

Find

$$\lim_{n \to \infty} n^n$$

Find

$$\lim_{n\to\infty}\frac{1}{n^n}$$

Find

$$\lim_{n\to\infty}\left(1-\left|\frac{\sin(n)}{n}\right|\right)$$

Find

$$\lim_{n\to\infty}\frac{n}{2^n}$$

Find

$$\lim_{n\to\infty}\frac{n^2}{2^n}$$

Find

$$\lim_{n \to \infty} \frac{n!}{n^n}$$

Find

$$\lim_{n\to\infty}\frac{n!}{2^n}$$

$+\infty$ and $-\infty$

1) They are ***not*** real numbers and do ***not*** necessarily obey the rules of arithmetic for real numbers.
2) We often act as if they do.
3) We need guidelines.

Add $+\infty$ and $-\infty$ to **R** and extend the ordering by

$$-\infty < a < +\infty$$

for every real number $a \in \mathbf{R} \cup \{+\infty, -\infty\}$.

$+\infty$ and $-\infty$

If $a \in \mathbf{R}$ then we define the following

1) $a + \infty = +\infty$
2) $a - \infty = -\infty$
3) If $a > 0$, then $a \times \infty = \infty$ and $a \times -\infty = -\infty$
4) If $a < 0$, then $a \times \infty = -\infty$ and $a \times -\infty = +\infty$

We may adopt the following conventions:

$$a/\infty = 0 \text{ and } a/(-\infty) = 0$$

Limits of Sequences

Limit of $\{a_n\}$ exists IFF we can compute L.

Will this always work?

Can we always find the limit?

Do we have to be able to find the limit as a number?

Theorem

<u>Theorem</u>: *Every convergent sequence is bounded.*

Is the converse true?

Is it true that every bounded sequence converges?

Find a proof or a counterexample.

Definitions

A sequence $\{a_n\}$ is ***increasing*** if $a_n \leq a_{n+1}$ for every n.

A sequence $\{a_n\}$ is ***decreasing*** if $a_n \geq a_{n+1}$ for every n.

A sequence is *monotone* (*monotonic*) if it is either increasing or decreasing.

Monotone Convergence Theorem

<u>Theorem</u>: *Every bounded monotonic sequence converges.*

<u>Proof</u>:

Let $\{a_n\}$ be a bounded increasing sequence and let $S = \{a_n \mid n \in N\}$. Since the sequence is bounded, $a_n < M$ for some real number M and for all n.

Therefore S is bounded and has a least upper bound. Let $u = \text{lub } S$ and let $\varepsilon > 0$.

Theorem

Since u=lub S and $\varepsilon > 0$, $u - \varepsilon$ is ***not*** an upper bound for S. Thus there is an integer K so that $a_K > u - \varepsilon$. Since $\{a_n\}$ is increasing then for all $n > K$, $a_n \geq a_N$ and for all $n > K$

$$u - \varepsilon < a_n \leq u.$$

Thus, $|\, a_n - u| < \varepsilon$ for all $n > K$ and lim $a_n = u =$ lub S.

Consequences

1) The decimal representation of a real number converges.

$$m < m.d_1d_2d_3d_4\ldots = m + \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \cdots \leq m+1$$

Let $a_n = m.d_1d_2d_3d_4\ldots d_n$. Then $a_n \leq a_{n+1}$ so $\{a_n\}$ is increasing.

2) Let $a_0 = 1$ and $a_{n+1} = 1/(1+ a_n)$

Consequences

2) Let $a_0 = 1$ and $a_{n+1} = 1 + \sqrt{a_n}$.

Does it converge? Is it monotone?

$a_0 = 1 \quad a_1 = 1 + \sqrt{a_0}) = 2$

$a_2 = 1 + \sqrt{a_1}) = 1 + \sqrt{2} \approx 2.4142...$

$a_3 = 1 + \sqrt{a_2} = 1 + \sqrt{2.4142...} \approx 2.55377...$

Prove it is increasing by induction on *n*.

Consequences

2) Let $a_0 = 1$ and $a_{n+1} = 1 + \sqrt{a_n}$.
Converges by Monotone Convergence Theorem. To what does it converge?
Assume: $\lim_{n\to\infty} a_n = L$
$a_{n+1} = 1 + \sqrt{a_n}$
$\lim_{n\to\infty} a_{n+1} = 1 + \lim_{n\to\infty} \sqrt{a_n}$
$L = 1 + \sqrt{(\lim_{n\to\infty} a_n)}$
$L = 1 + \sqrt{L}$
$(L - 1)^2 = L$ so $L^2 - 3L + 1 = 0$
$L = (3 \pm \sqrt{(9 - 4)})/2 = (3 \pm \sqrt{5})/2$
Which one is it? It cannot be both. Why?

Theorem

Theorem: Let $\{a_n\}$ be a sequence of real numbers.

(i) If $\{a_n\}$ is an unbounded monotonically increasing sequence, then $\lim a_n = +\infty$.

(ii) If $\{a_n\}$ is an unbounded monotonically decreasing sequence, then $\lim a_n = -\infty$.

Theorem

Theorem: Suppose that $\{a_n\}$ is a monotone increasing sequence and $\{b_n\}$ is a monotone decreasing sequence such that

$$a_n \leq b_n \text{ for all n = 0,1,2,...}$$

and

$$\{a_n - b_n\} \to 0$$

Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

Bolzano-Weierstrauss Theorem

Theorem: *Every sequence contains a monotone subsequence.*

Theorem: (Bolzano – Weierstrauss)
Every bounded sequence has a convergent subsequence.

The Cauchy Property

Definition: A sequence $\{a_n\}$ is said to have the Cauchy property if for every $\varepsilon > 0$ there is an index K so that if $n,m > K$ then

$$| a_m - a_n| < \varepsilon.$$

Theorems

Lemma:

Convergent sequences have the Cauchy property.

Theorem:

A sequence is a convergent sequence if and only if it has the Cauchy property.

Problems

Compute the limit if it exists:

$a_0 = 1$ and

$$a_{n+1} = \sqrt{a_n + \frac{1}{a_n}}$$

Problems

Compute the limit if it exists:

$$a_0 = 1 \text{ and}$$

$$a_{n+1} = 3 - \frac{1}{a_n}$$

Problems

Compute the limit if it exists:

$$a_0 = 0 \text{ and}$$

$$a_{n+1} = \frac{a_n + 1}{a_n + 2}$$

Problems

Compute the limit if it exists:

$$a_0 = 1 \text{ and}$$

$$a_{n+1} = \frac{a_n + 1}{a_n + 2}$$

Problems

Compute the limit if it exists:

$$a_0 = 0 \text{ and}$$

$$a_{n+1} = a_n{}^2 + \frac{1}{4}$$

Limits of Functions

How can we approach the definition of the limit of a function at a point in its domain?

Thus far, we have only defined the limit of a sequence. Can we make use of this?

Limits of Functions

Definition: Let $S \subset \boldsymbol{R}$, and let $a \in \boldsymbol{R}^* = \boldsymbol{R} \cup \{-\infty, \infty\}$ that is the limit of some sequence in S. Let $L \in \boldsymbol{R}^*$. We say that

$$\lim_{x \to a^S} f(x) = \mathrm{L}$$

if f is a function defined on S and for every sequence $\{x_n\} \subset S$, $\lim_{n\to\infty} x_n = a$, we have that $\lim_{n\to\infty} f(x_n) = L$.

Limits of Functions

What are some of the positive aspects of this definition?

What are some of the drawbacks of this definition?

Limits of Functions

Definition:

a) For $a \in \mathbf{R}$ and $f:\mathbf{R}\to\mathbf{R}$ we shall write

$$\lim_{x\to a} f(x) = L$$

provided $\lim_{x\to a}{}^{S} f(x) = L$ for some $S = J\setminus\{a\}$, where J is an open interval containing a.

$\lim_{x\to a} f(x) = L$ is the *two-side limit* of f at a.

Limits of Functions

b) For $a \in \mathbf{R}$ and $f : \mathbf{R} \to \mathbf{R}$ we shall write

$$\lim_{x \to a^+} f(x) = L$$

provided $\lim_{x \to a^S} f(x) = L$ for some open interval $S = (a,b)$.

$\lim_{x \to a^+} f(x) = L$ is the *right-hand limit* of f at a.

Limits of Functions

c) For $a \in \mathbf{R}$ and $f : \mathbf{R} \to \mathbf{R}$ we shall write

$$\lim_{x \to a^-} f(x) = L$$

provided $\lim_{x \to a^S} f(x) = L$ for some open interval $S = (c,a)$.

$\lim_{x \to a^-} f(x) = L$ is the *left-hand limit* of f at a.

Consequences for Functions

<u>Theorem</u>: Let f_1 and f_2 be functions for which the limits $\lim_{x\to a^S} f_1(x) = L_1$ and $\lim_{x\to a^S} f_2(x) = L_2$ exist and are finite. Then

1. $\lim_{x\to a^S} (f_1 + f_2)(x)$ exists and equals $L_1 + L_2$.
2. $\lim_{x\to a^S} (f_1 f_2)(x)$ exists and equals $L_1 L_2$.
3. $\lim_{x\to a^S} (f_1 / f_2)(x)$ exists and equals L_1/L_2 provided $L_2 \neq 0$ and $f_2(x) \neq 0$ for $x \in S$

Consequences for Functions

Theorem: Let f be a function for which the limit $\lim_{x\to a^S} f(x) = L$ and is finite. If g is a function defined on the set $\{f(x) | x \in S\} \cup \{L\}$, then $\lim_{x\to a^S} g \circ f(x)$ exists and equals $g(L)$.

Example

$$f(x) = 1 + x\sin\frac{\pi}{x}, x \neq 0 \qquad g(x) = \begin{cases} 4 & x \neq 1 \\ -4 & x = 1 \end{cases}$$

$$\lim_{x\to 0} f(x) = 1 \text{ and } \lim_{x\to 1} g(x) = 4$$

$$\lim_{x\to 0} g(f(x)) = ?$$

Is it 4? Are you sure?

Example

Let $x_n = \dfrac{2}{n}$

$$f(x_n) = 1 + \frac{2}{n}\sin\frac{n\pi}{2} = \begin{cases} 1 & \text{if } n \text{ is even} \\ 1 \pm \dfrac{2}{n} \neq 1 & \text{if } n \text{ is odd} \end{cases}$$

$$g(f(x_n)) = \begin{cases} -4 & \text{if } n \text{ is even} \\ 4 & \text{if } n \text{ is odd} \end{cases}$$

$$\lim_{n\to\infty} x_n = 0, \text{ but } \lim_{x\to 0} g(f(x)) \text{ does not exist.}$$

More Results on Limits

<u>Theorem</u>: Let f be a function defined on $S \subseteq \mathbf{R}$, let $a \in \mathbf{R}$ be a limit of some sequence in S, and let $L \in \mathbf{R}$. Then the limit $\lim_{x \to a^S} f(x) = L$ if and only if for each $\varepsilon > 0$ there exists a $\delta > 0$ so that if $x \in S$ and $|x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

<u>Corollary 1</u>: Let f be a function defined on $J \setminus \{a\}$ for some open interval J containing a and let $L \in \mathbf{R}$. Then $\lim_{x \to a} f(x) = L$ if and only if for each $\varepsilon > 0$ there exists a $\delta > 0$ so that if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

More Results on Limits

Corollary 2: Let f be a function defined on (a,b) and let $L \in \mathbf{R}$. Then the limit $\lim_{x\to a^+} f(x) = L$ if and only if for each $\varepsilon > 0$ there exists a $\delta > 0$ so that if $a < x < a + \delta$ then $|f(x) - L| < \varepsilon$.

Theorem 2: Let f be a function defined on $J \setminus \{a\}$ for some open interval J containing a. Then $\lim_{x\to a} f(x)$ exists if and only if the limits $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ both exist and are equal, in which case all three limits are equal.

Operational Results for Limits

Let $a \in \boldsymbol{R}^* = \boldsymbol{R} \cup \{-\infty, \infty\}$; $L, M \in \mathbf{R}$; and let f and g be functions. Let $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$. Then if c is a constant,

a) $\lim_{x \to a} c = c$;
b) $\lim_{x \to a} [cf(x)] = cL$;
c) $\lim_{x \to a} [f(x) + g(x)] = L + M$;
d) $\lim_{x \to a} [f(x)g(x)] = LM$;
e) $\lim_{x \to a} [f(x)/g(x)] = L/M$ if $M \neq 0$;
f) $\lim_{x \to a} x = a$;
g) $\lim_{x \to a} x^n = a^n$;
h) $\lim_{x \to a} x^{1/n} = a^{1/n}$, if $a^{1/n}$ is defined.