MATH 6102
Spring 2009

Continuous Functions

Sequential Continuity

Let f be a real-valued function whose domain is a subset of R. The function *f is continuous at $x = a$* if, for every sequence of real numbers $\{x_n\} \subset \text{dom}(f)$ that converges to a, we have that

$$\lim_{n\to\infty} f(x_n) = f(a).$$

If f is continuous at each point of a set $S \subset \text{dom}(f)$, then we say that f is continuous on S. The function f is said to be continuous if it is continuous on dom(f).

Discussion of Last Statement

The function f is said to be continuous if it is continuous on dom(f).

Question:

Is the function $f(x) = 1/x$ continuous on the interval $(-1, 1)$?

Alternate definition: (Stewart) A function f is continuous on an interval if it is continuous at every number in that interval.

Discussion of Last Statement

Alternate definition: (Stewart) A function f is continuous on an interval if it is continuous at every number in that interval.

Question:

Is the function $f(x) = 1/x$ *continuous on the interval* $(-1, 1)$ *?*

Discussion of Last Statement

By convention we will define the domain of a function to be the largest subset of the real numbers on which f is defined.

Should we ask if $f(x) = \sqrt{x^2 - 1}$ is continuous at $x = 0$?

Discontinuities

If a function $f(x)$ is not continuous at $x = a$, we may say that it is *discontinuous* at $x = a$ or that it has a *discontinuity* at $x = a$.

Does $f(x) = 1/x$ have a discontinuity at $x = 0$?

Let $F(x) = \begin{cases} 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Does $F(x)$ have a discontinuity at $x = 0$?

Discontinuities

Removable discontinuity

1) $\lim_{x \to a^+} f(x) = L < \infty$

2) $\lim_{x \to a^-} f(x) = L < \infty$

3) $f(a) \neq L$,

a is called a **removable discontinuity**.

This discontinuity can be *removed* by redefining the function so that f is continuous at a.

$$g(x) = \begin{cases} f(x) & \text{if } x \neq a \\ L & \text{if } x = a \end{cases}$$

Discontinuities

The limits L^- and L^+ exist and are finite, but not equal. Then, x_0 is called a **jump discontinuity** or **step discontinuity**. For this type of discontinuity, the value of $f(x_0)$ does not matter.

$$\lim_{x \to a^+} f(x) = L^+ \neq L^- = \lim_{x \to a^-} f(x)$$

Discontinuities

One or both of the limits L^- and L^+ does not exist or is infinite. Then, x_0 is called an **essential discontinuity**, or **infinite discontinuity**.

Another Definition

Let f be a real-valued function whose domain is a subset of R. Then f is continuous at $a \in \mathrm{dom}(f)$ if and only if for each $\varepsilon > 0$ there exists $\delta > 0$ so that if $x \in \mathrm{dom}(f)$ and $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$.

This is known as the ε-δ definition of continuity. This is essentially Dirichlet's definition and is rigorous with respect to the definition of limits.

A Working Definition

A function f is said to be continuous at a point $x = a$ if

1. $f(a)$ exists (is defined),
2. $\lim_{x \to a} f(x)$ exists, and
3. $\lim_{x \to a} f(x) = f(a)$.

All of these definitions are equivalent!

Example

Let $f(x) = 3x^2 + 2x - 1$ *for* $x \in R$. Prove that f is continuous on R

By

1. using the limit definition,
2. using the ε-δ definition,
3. using the last definition.

Example by 1

Let $a \in R$, and let $\{x_n\}$ be any sequence converging to a; *i.e.* $\lim_{x\to a} x_n = a$. Then

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty}(3x_n^2 + 2x_n - 1)$$
$$= 3\lim_{n\to\infty} x_n^2 + 2\lim_{n\to\infty} x_n - 1$$
$$= 3\left(\lim_{n\to\infty} x_n\right)^2 + 2\lim_{n\to\infty} x_n - 1 = 3a^2 + 2a - 1 = f(a)$$

Example by 2

Let $\varepsilon > 0$ be given and let $a \in R$. Then

$$|f(x) - f(a)| = |(3x^2 + 2x - 1) - (3a^2 + 2a - 1)|$$
$$= |(3x^2 - 3a^2) + (2x - 2a)|$$
$$= 3|x-a| \bullet |x+a| + 2|x-a|$$
$$= (3|x+a| + 2)|x-a|$$

To see how big this can get, in terms of $|x - a|$, we need a bound on $|x + a|$ that does NOT depend on x.

Example by 2

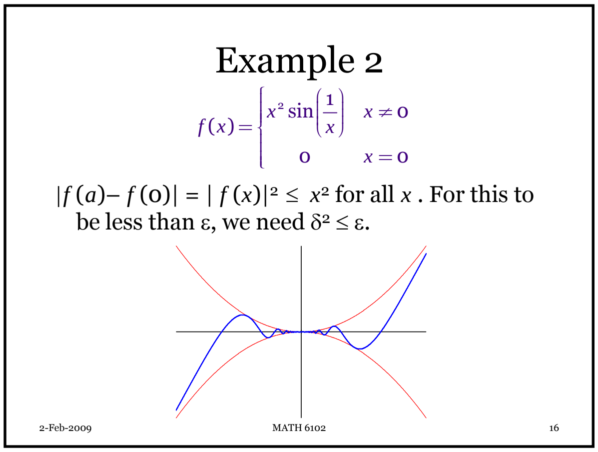
If $|x - a| < 1$, then $|x| < |a| + 1$ and

$$|x + a| < |x| + |a| < 2|a| + 1$$

$$|f(x) - f(a)| = (3|x+a| + 2)|x-a|$$
$$< (3(2|a|+1) + 2)|x-a|$$
$$= (6|a|+5)|x-a|$$

For this to be $< \varepsilon$, $|x - a|$ must be less than 1 and less than $\varepsilon/(6|a|+5)$. So take,

$$\delta < \min\left\{1, \frac{\epsilon}{(6|a|+5)}\right\}$$



Example 2

$$f(x)=\begin{cases} x^2\sin\left(\frac{1}{x}\right) & x\neq 0 \\ 0 & x=0 \end{cases}$$

$|f(a)-f(0)| = |f(x)|^2 \leq x^2$ for all x. For this to be less than ε, we need $\delta^2 \leq \varepsilon$.

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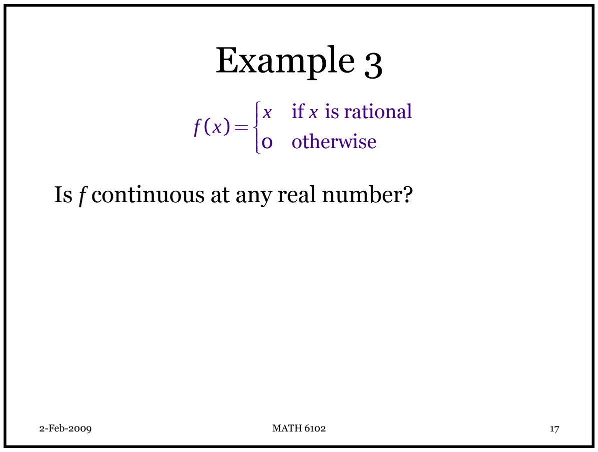

Example 3

$$f(x)=\begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{otherwise} \end{cases}$$

Is f continuous at any real number?

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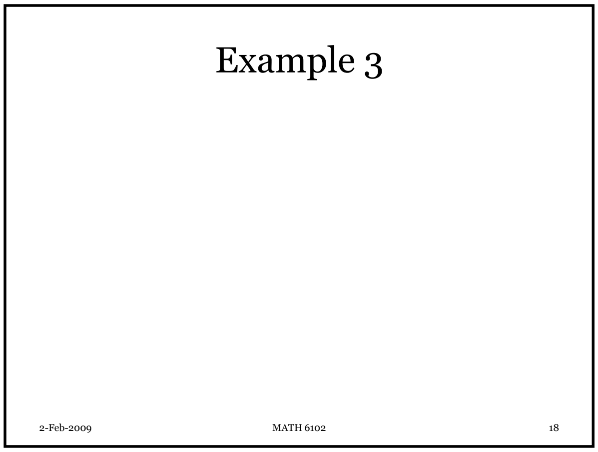

Example 3

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Example 4

For each rational number x, write x as p/q where p and q are integers with no common factors and $q>0$. Define the function g by

$$g(x)=\begin{cases}\frac{1}{q} & \text{if } x \text{ is rational}\\ 0 & \text{otherwise}\end{cases}$$

Thus, $g(x)=1$ for all integers,

$g(1/2)=g(-1/2)=g(3/2)=1/2$.

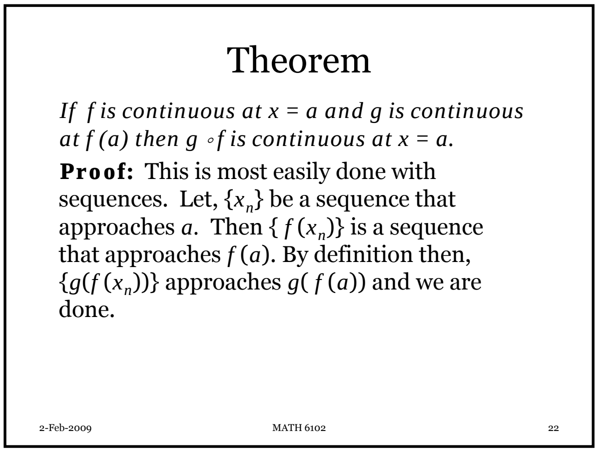
Show that g is continuous at each irrational and discontinuous at each rational.

Example 4

Consequences

Let f and g be continuous at $x = a$. Then all of the following are continuous at $x = a$

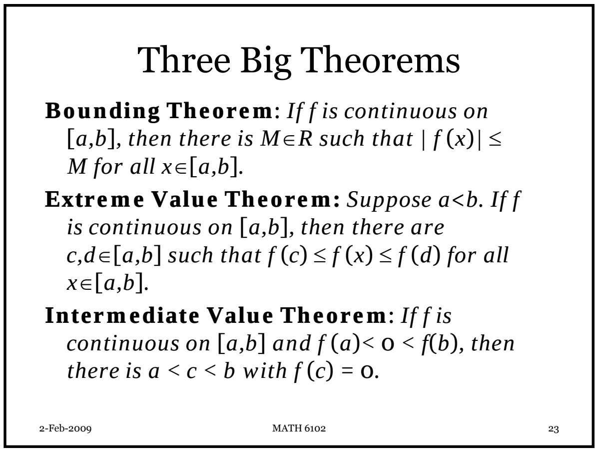
i) $|f|$,
ii) kf,
iii) $f \pm g$,
iv) $f \times g$,
v) f/g if $g(a) \neq 0$,
vi) $f^{p/q}$ if $f > 0$,



Theorem

If f is continuous at $x = a$ and g is continuous at $f(a)$ then $g \circ f$ is continuous at $x = a$.

Proof: This is most easily done with sequences. Let, $\{x_n\}$ be a sequence that approaches a. Then $\{f(x_n)\}$ is a sequence that approaches $f(a)$. By definition then, $\{g(f(x_n))\}$ approaches $g(f(a))$ and we are done.

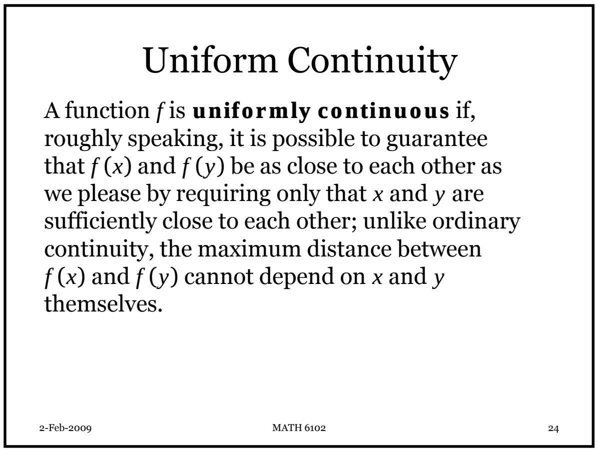


Three Big Theorems

Bounding Theorem: *If f is continuous on $[a,b]$, then there is $M \in R$ such that $|f(x)| \leq M$ for all $x \in [a,b]$.*

Extreme Value Theorem: *Suppose $a<b$. If f is continuous on $[a,b]$, then there are $c,d \in [a,b]$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in [a,b]$.*

Intermediate Value Theorem: *If f is continuous on $[a,b]$ and $f(a) < 0 < f(b)$, then there is $a < c < b$ with $f(c) = 0$.*



Uniform Continuity

A function f is **uniformly continuous** if, roughly speaking, it is possible to guarantee that $f(x)$ and $f(y)$ be as close to each other as we please by requiring only that x and y are sufficiently close to each other; unlike ordinary continuity, the maximum distance between $f(x)$ and $f(y)$ cannot depend on x and y themselves.

Uniform Continuity

Continuity itself is a ***local*** (*pointwise*) property of a function — that is, a function *f* is continuous, or not, at a particular point. When we speak of a function being continuous on an interval, we mean only that it is continuous at each point of the interval.

In contrast, uniform continuity is a ***global*** property of *f*, in the sense that the standard definition refers to *pairs* of points rather than individual points.

Uniform Continuity

Definition: A function *f* defined on a set *D* of real numbers is said to be uniformly continuous on *D* if for every $\varepsilon > 0$ there is a corresponding $\delta > 0$ such that whenever $u, v \in D$ such that $|v - u| < \delta$ it follows that $|f(v) - f(u)| < \varepsilon$.

Continuity

Pointwise Continuous

$$(\forall u \in D)(\forall \epsilon > 0)(\exists \delta > 0)(\forall v \in D)[(|v - u| < \delta) \Rightarrow (|f(v) - f(u)| < \epsilon)]$$

In this instance first select a point $u \in D$ and $\varepsilon > 0$. Then ***find*** $\delta > 0$ (which may depend upon both *u* and δ) in such a way that every $v \in D$ with a certain property satisfies a certain inequality.

Note the emphasis: *may depend upon both u and δ.*

Continuity

Uniformly Continuous

$$(\forall\epsilon > 0)(\exists\delta > 0)(\forall u \in D)(\forall v \in D)[(|v-u| < \delta) \Rightarrow (|f(v)-f(u)| < \epsilon)]$$

In this case, we choose ε, and ε only, first. Then find δ > 0 in such a way that every pair of numbers *u*,*v* that are within δ of each other satisfy a certain inequality. Here δ may depend upon ε only, and not upon both of the numbers *u* and ε.

Continuity

To show pointwise continuity on *D*, we can choose different δ's in different parts of *D*, even though ε has not changed.

To show uniform continuity on *D*, once ε has been specified, you must find a single δ and show that it works everywhere in *D*.

Continuous but not Uniform

Consider the function $x \to x^2$. We know that this is a continuous function on *R*.

Claim: *f is not uniformly continuous on R*.

$$(\forall\epsilon > 0)(\exists\delta > 0)(\forall u \in D)(\forall v \in D)[(|v-u| < \delta) \Rightarrow (|f(v)-f(u)| < \epsilon)]$$

The negation is:

$$(\exists\epsilon > 0)(\forall\delta > 0)(\exists u \in D)(\exists v \in D)[(|v-u| < \delta) \wedge (|f(v)-f(u)| \geq \epsilon)]$$

So we want to show that if ε=1, then for any δ>0 it is possible to find *u* and *v* with $|u - v| < \delta$, but $|u^2 - v^2| > \varepsilon$

Continuous but not Uniformly

Let $\delta > 0$ be given. Now, let $u = 1/\delta$ and let $v = u + \delta/2$.

Now, $|v - u| = \delta/2 < \delta$. What about $u^2 - v^2$?

$$\begin{aligned} \left|v^2 - u^2\right| &= \left|\left(\frac{1}{\delta} + \frac{\delta}{2}\right)^2 - \left(\frac{1}{\delta}\right)^2\right| \\ &= \left|\frac{1}{\delta^2} + 1 + \frac{\delta^2}{4} - \frac{1}{\delta^2}\right| \\ &= 1 + \frac{\delta^2}{4} > 1 = \epsilon \end{aligned}$$

Continuous but not Uniformly

Should this be a surprise? What does the squaring function do? Can you think of a function that might be uniformly continuous on R?

Uniform Continuity

Theorem: Every uniformly continuous function is pointwise continuous.

We know the converse is not true, but what condition would make the converse true?

Uniform Continuity

Theorem: If f is a continuous function on $[a,b]$, then f is uniformly continuous on $[a,b]$.

Theorem: If f is a continuous function on $[0,\infty)$ and $\lim_{x\to\infty} f(x)$ exists (and is finite), then f is uniformly continuous.

Continuity and Inverse Functions

A function f defined on an interval I is *increasing* on I if $u < v$ for $u,v \in I$ we have that $f(u) \le f(v)$. f is *strictly increasing* on I if u < v for $u,v \in I$ we have that $f(u) < f(v)$. We say that f is *decreasing* on I if $u < v$ for $u,v \in I$ we have that $f(u) \ge f(v)$. If the same conditions imply that $f(u) > f(v)$, we call f *strictly decreasing* on I. We use the term *(strictly) monotonic* of a function to mean that that function is *(strictly) increasing* on I or that it is *(strictly) decreasing* on I.

One-to-one and Onto Functions

A function f, defined on some domain $A \subseteq$ R, and taking its values in some set $B \subseteq R$ is said to be *onto* if for every $y \in B$ there is an $x \in A$ such that $f(x) = y$.

A function f, defined on some domain $A \subseteq$ R, and taking its values in some set $B \subseteq R$ is said to be *one-to-one* on A if for any x_1, $x_2 \in A$ with $x_1 \ne x_2$ then we have that $f(x_1) \ne f(x_2)$.

One-to-one and Onto Functions

Give an example of a function that is onto and an example of a function that is not onto.

Give an example of a one-to-one function and a function that is not one-to-one.

onto = surjective
one-to-one = injective

Inverse Functions

Let A, $B \subseteq R$, and let $f : A \to R$ and $g : B \to R$ be functions. If $f(g(y)) = y$ for every $y \in B$ and $g(f(x)) = x$ for every $x \in A$, then we say that f and g are inverses to each other.

We will write $f^{-1} = g$ or $g^{-1} = f$.

Theorem: A function $f : A \to R$ has an inverse whose domain is $B \subseteq R$ if and only if f is one-to-one and onto.

Continuous Inverse: Suppose that $f: (a,b) \to (c,d)$ is strictly monotonic and onto. Then f is continuous on (a,b) and has an inverse $f^{-1}: (c,d) \to (a,b)$ which is also strictly monotonic, onto, and continuous.