MATH 6102
Spring 2009

Families of Functions

How is this like building numbers?

We build our number systems from the integers. What nice properties do the integers have? What are especially helpful when studying rational numbers and real numbers?

1) Prime numbers
2) Unique factorization
3) Add and multiply – it forms a *commutative ring*
4) Euclidean algorithm

Where should we start building functions?

The simplest function is the constant function.

The next is a linear function: $f(x) = ax + b$

What is the next function in our study chain?

Quadratic functions: $f(x) = ax^2 + bx + c$

What is the complete set?

Polynomials

(Polynomials over Q)

What is a polynomial?

Definition:

Polynomials

Is $x + x^{-1}$ a polynomial?

Is $f(x) = \sqrt{x^2 - 1}$ a polynomial?

Is $x/2$ a polynomial?

Is x^{-2} a polynomial?

Is $x^{6/2}$ a polynomial?

Polynomials

For given numerical constants a_n, $n > 0$, in some field (often Q, R or C), with a_n non-zero, a polynomial (function) of degree n is a function of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots a_n x^n$$

or

$$p(x) = \sum_{k=0}^{n} a_k x^k$$

Polynomial Properties

1. Are there prime polynomials?
2. Does it depend on the base field?
3. Can you factor x^2+1 over the rationals?
   Can you factor $x^2 - 4$ over the rationals?
   Can you factor $x^2 - 6$ over the rationals?
4. Can you factor $x^2 - 1$ over the rationals? Can you do it in more than one way?

Polynomial Properties

Polynomials that cannot be factored and have leading coefficient 1 are called monic, irreducible polynomials. They play the role of prime numbers when we look at the set of polynomials with rational coefficients.

What are the *irreducible* polynomials when we look at the set of polynomials with ***real*** coefficients?

The graph of a function

The graph of a function, $f:\mathbf{R} \to \mathbf{R}$, is the set of order pairs

$$\Gamma_f = \{(x,y) \in \mathbf{R}^2 \mid y = f(x)\}$$

If f is a continuous function then Γ_f is a connected set in the plane.

We know that we can learn a lot about a function by looking at it's graph.

The graph of a polynomial

What are the important features of the graph of a polynomial?

1)
2)
3)
4)
5)

The graph of a polynomial

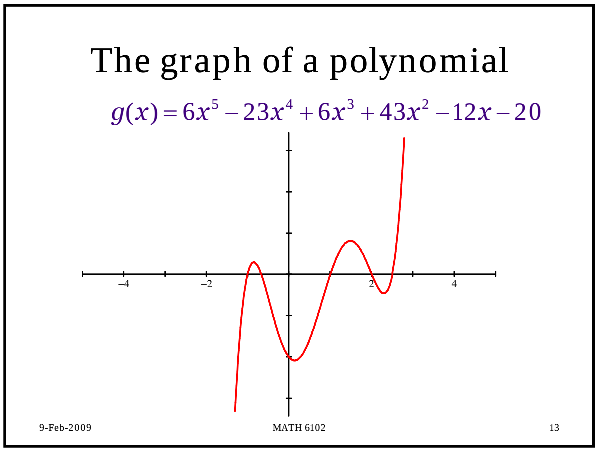
What are the important features of the graph of a polynomial?

1) turning points and number
2) x-intercepts (*aka* roots) and number
3) y-intercept
4) end behavior

The graph of a polynomial

Let $g(x) = 6x^5 - 23x^4 + 6x^3 + 43x^2 - 12x - 20$

1) How many roots could it have?
2) How many turning points could it have?
3) What is the y-intercept?
4) What is its end behavior?



The graph of a polynomial

$g(x) = 6x^5 - 23x^4 + 6x^3 + 43x^2 - 12x - 20$

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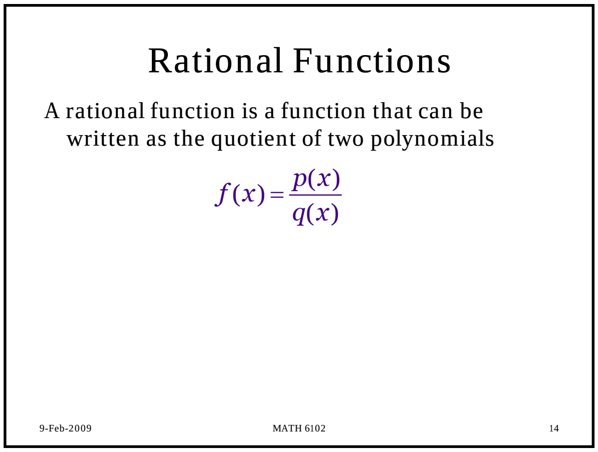

Rational Functions

A rational function is a function that can be written as the quotient of two polynomials

$$f(x) = \frac{p(x)}{q(x)}$$

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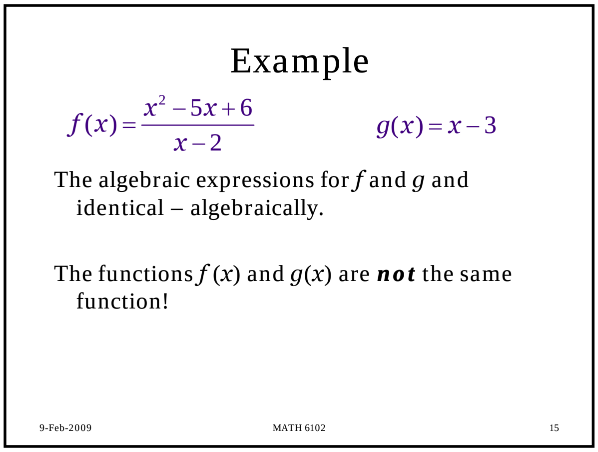

Example

$f(x) = \dfrac{x^2 - 5x + 6}{x - 2}$ $\qquad g(x) = x - 3$

The algebraic expressions for f and g and identical – algebraically.

The functions $f(x)$ and $g(x)$ are ***not*** the same function!

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The graph of a rational function

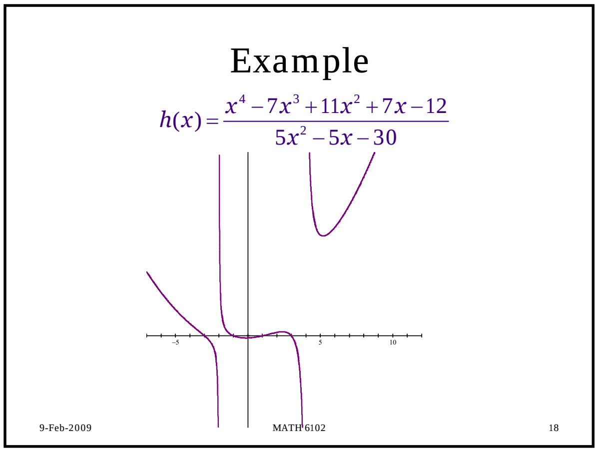
What are the important features of the graph of a rational function?

1) roots of the denominator and number
2) x-intercepts (*aka* roots) and number
3) y-intercept
4) turning points
5) end behavior

Example

$$h(x)=\frac{x^4-7x^3+11x^2+7x-12}{5x^2-5x-30}$$

1) What is the domain?
2) What are the x-intercepts?
3) What are the turning points? Does the denominator play a factor in determining the turning points?
4) What is the end behavior?
5) What is the behavior "in the large"?



Example

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Algebraic Functions

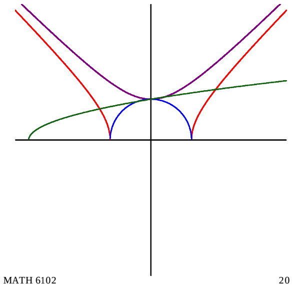
These are functions involving roots in addition to being rational functions. The basic examples are:

$f(x)=\sqrt{x^2+a^2}$

$f(x)=\sqrt{x^2-a^2}$

$f(x)=\sqrt{a^2-x^2}$

$f(x)=\sqrt{ax+b}$

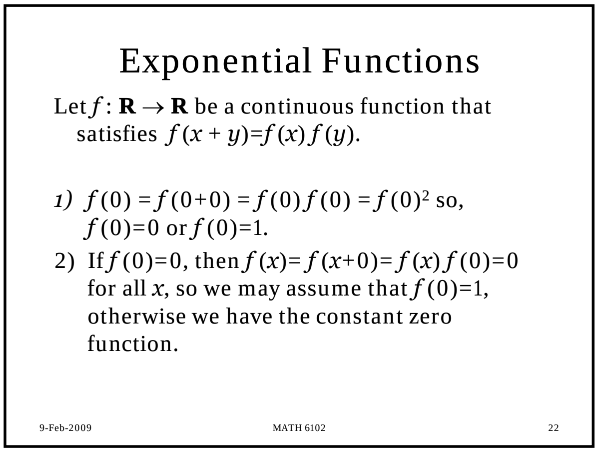


Transcendental Functions

These are trigonometric functions, exponential functions, logarithmic functions and special functions.

We will consider hyperbolic trigonometric functions and generalized logarithms next time.

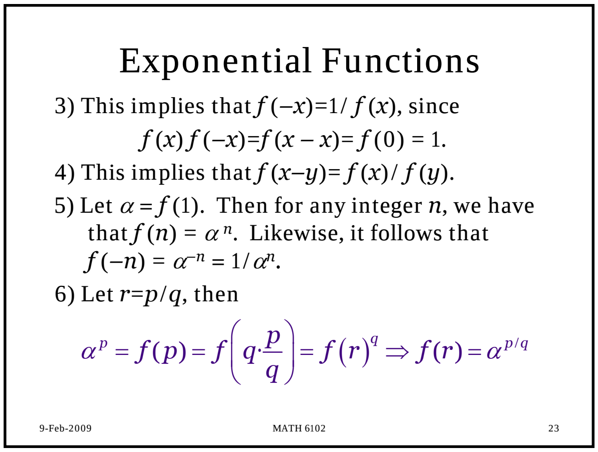
First, exponential functions.



Exponential Functions

Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function that satisfies $f(x + y) = f(x)f(y)$.

1) $f(0) = f(0+0) = f(0)f(0) = f(0)^2$ so, $f(0)=0$ or $f(0)=1$.
2) If $f(0)=0$, then $f(x)=f(x+0)=f(x)f(0)=0$ for all x, so we may assume that $f(0)=1$, otherwise we have the constant zero function.



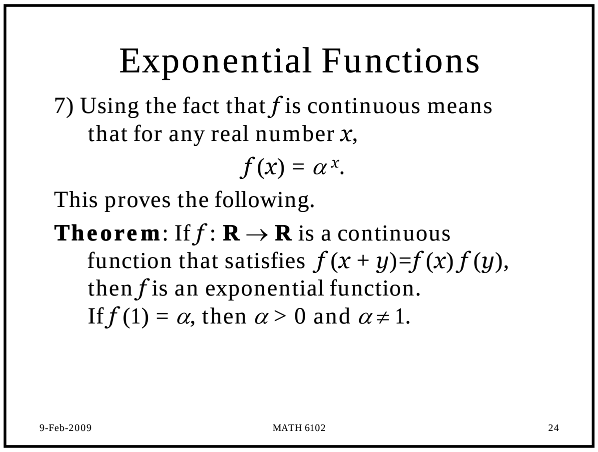
Exponential Functions

3) This implies that $f(-x)=1/f(x)$, since

$$f(x)f(-x)=f(x-x)=f(0) = 1.$$

4) This implies that $f(x-y)=f(x)/f(y)$.
5) Let $\alpha = f(1)$. Then for any integer n, we have that $f(n) = \alpha^n$. Likewise, it follows that $f(-n) = \alpha^{-n} = 1/\alpha^n$.
6) Let $r=p/q$, then

$$\alpha^p = f(p) = f\left(q\cdot\frac{p}{q}\right) = f(r)^q \Rightarrow f(r) = \alpha^{p/q}$$



Exponential Functions

7) Using the fact that f is continuous means that for any real number x,

$$f(x) = \alpha^x.$$

This proves the following.

Theorem: If $f : \mathbf{R} \to \mathbf{R}$ is a continuous function that satisfies $f(x + y)=f(x)f(y)$, then f is an exponential function. If $f(1) = \alpha$, then $\alpha > 0$ and $\alpha \neq 1$.

Exponential Functions

If $f : \mathbf{R} \to \mathbf{R}$ is a continuous function that satisfies $f(xy) = f(x) + f(y)$, then f is what type of function?

Logarithmic Functions

If b is a real number, $b > 0$ and $b \neq 1$, then the logarithm base b is defined by the equation:

$$\log_b(x) = y \text{ if and only if } b^y = x.$$

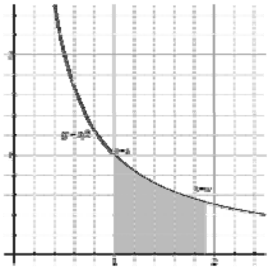
This requires that we have an exponential function defined first. We had to assume that the exponential function was continuous. Is there another way to do this?

Logarithms *via* Calculus

Consider the curve $y = 1/t$.

$$L(x) = \int_1^x \frac{1}{t}\,dt$$

1) $L(1) = 0$.



Logarithms *via* Calculus

Assume that $1 < a < b$. Then $a < ab$.

Claim:

$$\int_a^{ab} \frac{dt}{t} = \int_1^b \frac{dt}{t} = L(b)$$

This is a “simple substitution.”

Let $x = t/a$, then $t = ax$ and $dt = adx$.
Likewise, when $t = a$ then $x = 1$ and when $t = ab$ then $x = b$.

$$\int_a^{ab} \frac{dt}{t} = \int_1^b \frac{adx}{ax} = \int_1^b \frac{dx}{x} = L(b)$$

Logarithms *via* Calculus

From this we have the following.

Theorem: $L(ab) = L(a) + L(b)$

$$\begin{aligned} L(ab) &= \int_1^{ab} \frac{dt}{t} \\ &= \int_1^a \frac{dt}{t} + \int_a^{ab} \frac{dt}{t} \\ &= \int_1^a \frac{dt}{t} + \int_1^b \frac{dt}{t} \\ &= L(a) + L(b) \end{aligned}$$

Logarithms *via* Calculus

Theorem: $L(1/b) = -L(b)$.

Let $b > 1$.

$$\begin{aligned} L(1/b) &= \int_1^{1/b} \frac{dt}{t} \\ &= -\int_{1/b}^{1} \frac{dt}{t} \\ &= -\int_1^b \frac{dt}{t} \\ &= -L(b) \end{aligned}$$

Theorem: $L(a/b) = L(a) - L(b)$.

Logarithms *via* Calculus

Theorem: $L(a^b) = bL(a)$.

$$L(a^b) = \int_1^{a^b} \frac{dt}{t}.\ \text{Let } t = x^b.\quad dt = bx^{b-1}.$$
$$= \int_1^a \frac{bx^{b-1}dx}{x}$$
$$= b\int_1^a \frac{dx}{x}$$
$$= bL(a)$$

Logarithms *via* Calculus

Therefore, we have a function $L(x)$ that is defined on $(0,\infty)$ that satisfies:

- $L(1) = 0$,
- $L(ab) = L(a) + L(b)$,
- $L(a/b) = L(a) - L(b)$,
- $L(a^b) = b\,L(a)$,
- $L(x) < 0$, for $x < 1$,
- $L(x) > 0$, for $x > 1$,
- If $x < y$, then $L(x) < L(y)$

Logarithms *via* Calculus

Let e be the unique real number so that $L(e) = 1$. By approximation, we can show that $2 < e < 3$.

Define a new function, $E(x)$, by $x = E(y)$ if and only if $y = L(x)$. Note that

dom(E) = all reals

range(E) = positive reals

The antilog function

We have:

- $E(0) = 1$, $E(1)=e$,
- $E(a + b) = E(a) \times E(b)$,
- $E(a - b) = E(a)/E(b)$,
- $E(ab) = E(a)^b$,
- $E(x) < 1$, for $x < 0$,
- $E(x) > 1$, for $x > 0$,
- If $x < y$, then $E(x) < E(y)$
- Since $E(1)=e$, we have $E(x)=e^x$.

The antilog function

To find the derivative, we have:

$$\begin{aligned} 1 &= \frac{dx}{dx} \\ &= \frac{d}{dx} L(E(x)) \\ &= \frac{1}{E(x)} \frac{d}{dx} E(x) \end{aligned}$$

$$\frac{d}{dx} E(x) = E(x)$$

Old Uses for Logs

	0.000	0.001	0.002	0.003	0.004
1.250	0.09691001	0.09725731	0.09760433	0.09795107	0.09829754
1.255	0.09864373	0.09898964	0.09933528	0.09968064	0.10002573
1.260	0.10037055	0.10071509	0.10105935	0.10140335	0.10174707
1.265	0.10209053	0.10243371	0.10277661	0.10311925	0.10346162
1.270	0.10380372	0.10414555	0.10448711	0.10482840	0.10516943
1.275	0.10551018	0.10585067	0.10619090	0.10653085	0.10687054
1.280	0.10720997	0.10754913	0.10788803	0.10822666	0.10856502

log(1.277) = 0.10619090

log(1.283) = ?

Interpolation

	0.000	0.001	0.002	0.003	0.004
1.275	0.10551018	0.10585067	0.10619090	0.10653085	0.10687054

log(1.2773) = ?
log(1.277) = 0.10619090
log(1.278) = 0.10653085

$$\left.\begin{array}{l}\left.\begin{array}{l}1.277\\1.2773\end{array}\right\}0.0003\\1.278\end{array}\right\}0.001 \qquad 0.00033995\left\{\begin{array}{l}?\left\{\begin{array}{l}0.10619090\\ \log(1.2773)\end{array}\right.\\0.10653085\end{array}\right.$$

Interpolation

So, using the line segment that joins the two points we see that

$$\frac{0.0003}{0.001}=\frac{\log(1.2773)-0.10619090}{0.10653085-0.10619090}$$

$$\log(1.2773)=0.106292885$$

Multiplication/Division

$$B=\frac{23.44\times 34.92}{734.5\times 9.221}$$

$$\begin{aligned}\log(B)&=\log(23.44)+\log(34.92)-(\log(734.5)+\log(9.221))\\&=1.36995761+1.54307424-(2.86599180+0.96477802)\\&=2.91303185-3.83076982=-0.91773797\\&=-1+0.08226203\end{aligned}$$

$$\operatorname{antilog}(0.08226203)=1.208+\left(\frac{0.01\left(0.08242630-0.08226203\right)}{0.08242630-0.08206693}\right)$$

$$=1.20854$$

So $B=0.120854$

Calculating Logs

Taylor series:
$$\ln(x)=\sum_{n=1}^{\infty}-\frac{(1-x)^n}{n}$$

Modified:
$$\ln(1+x)=x\left(\frac{1}{1}-x\left(\frac{1}{2}-x\left(\frac{1}{3}-x\left(\frac{1}{4}-x\left(\frac{1}{5}-\ldots\right)\right)\right)\right)\right)$$

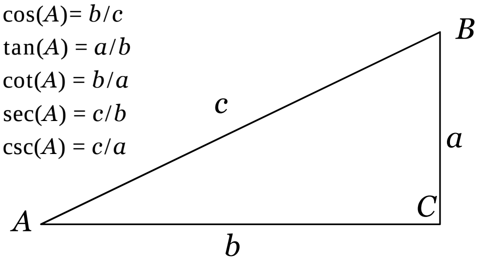
Modified again: $z=(x-1)/(x+1)$
$$\ln(x)=\ln\left(\frac{1+z}{1-z}\right)=2z\left(\frac{1}{1}+z^2\left(\frac{1}{3}+z^2\left(\frac{1}{5}+z^2\left(\frac{1}{7}+z^2\left(\frac{1}{9}+\ldots\right)\right)\right)\right)\right)$$

Calculating Logs

Let's find ln(1.o1). From this we can find log(1.01).
If x = 1.01, then
z = (1.01-1)/(1.01+1)=0.01/2.01 = 0.004975.
First iteration: ln(1.01) ≈ $2z(1)$
ln(1.01) ≈ 0.009950248756219.
Second iteration: ln(1.01) ≈ $2z(1+z^2/3)$
ln(1.01) ≈ 0.0099503308519493.
The TI-*n*spire gives us
ln(1.01) = 0.0099503308531681.

Trigonometric Functions

$\sin(A) = a/c$
$\cos(A)= b/c$
$\tan(A) = a/b$
$\cot(A) = b/a$
$\sec(A) = c/b$
$\csc(A) = c/a$



Trigonometric Properties

Law of Sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

Law of Cosines:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

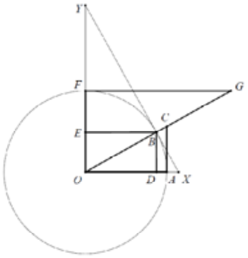
Law of Tangents

$$\frac{a+b}{a-b} = \frac{\tan\left[\frac{1}{2}(A+B)\right]}{\tan\left[\frac{1}{2}(A-B)\right]}$$

Associated Segments

$\angle BOD = \theta$

Function	**Segment**
$\sin\theta$	$BD = OE$
$\cos\theta$	$OD = BE$
$\tan\theta$	AC
$\cot\theta$	GF
$\sec\theta$	OC
$\csc\theta$	OG



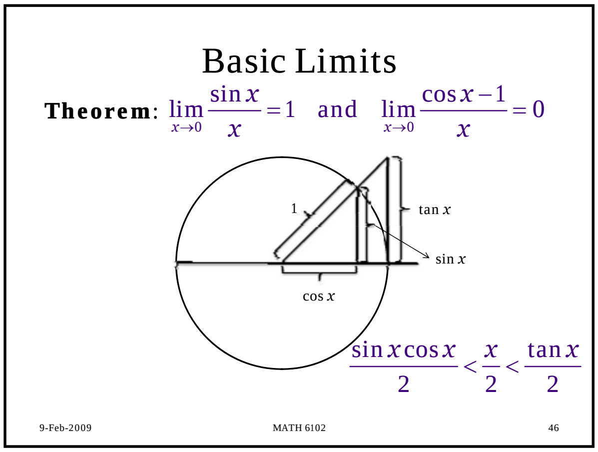
Functional Properties

there exists exactly one pair of real functions s and c such that for all real numbers x and y, the following equations hold:

$$[s(x)]^2 + [c(x)]^2 = 1$$

$$s(x+y) = s(x)c(y) + c(x)s(y)$$

$$c(x+y) = c(x)c(y) - s(x)s(y)$$



Basic Limits

Theorem: $\lim_{x\to 0}\frac{\sin x}{x}=1$ and $\lim_{x\to 0}\frac{\cos x-1}{x}=0$

$$\frac{\sin x\cos x}{2}<\frac{x}{2}<\frac{\tan x}{2}$$

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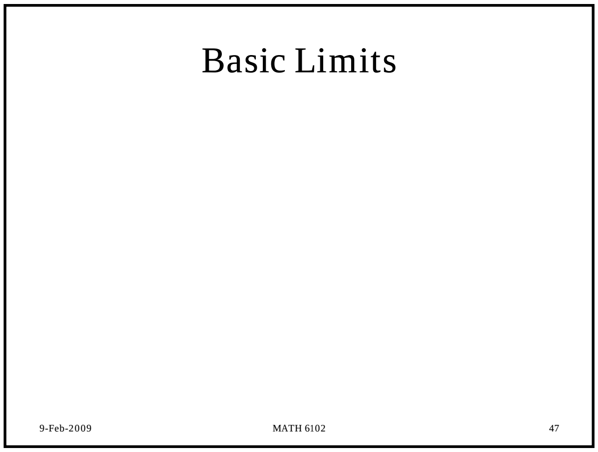

Basic Limits

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