MATH 6102

Spring 2009

A Bestiary of Calculus Special Functions

Transcendental Functions

Last time we discussed exponential, logarithmic, and trigonometric functions.

Theorem 1: If $f : \mathbf{R} \to \mathbf{R}$ is a continuous function that satisfies $f(x + y) = f(x)f(y)$, then f is an exponential function. If $f(1) = \alpha$, then $\alpha > 0$ and $\alpha \neq 1$.

Theorem 2: If $f : \mathbf{R}^{+} \to \mathbf{R}$ is a continuous function that satisfies $g(xy) = g(x) + g(y)$, then g is an logarithmic function. If $g(b) = 1$, then $b > 0$ and $b \neq 1$.

Transcendental Functions

Theorem 3: There exists exactly one pair of real functions s and c such that for all real numbers x and y, the following equations hold:

$$[s(x)]^2 + [c(x)]^2 = 1$$

$$s(x+y) = s(x)c(y) + c(x)s(y)$$

$$c(x+y) = c(x)c(y) - s(x)s(y)$$

Transcendental Functions

Theorem 4: There exists exactly one pair of real functions *s* and *c* such that for all real numbers *x* and *y*, the following equations hold:

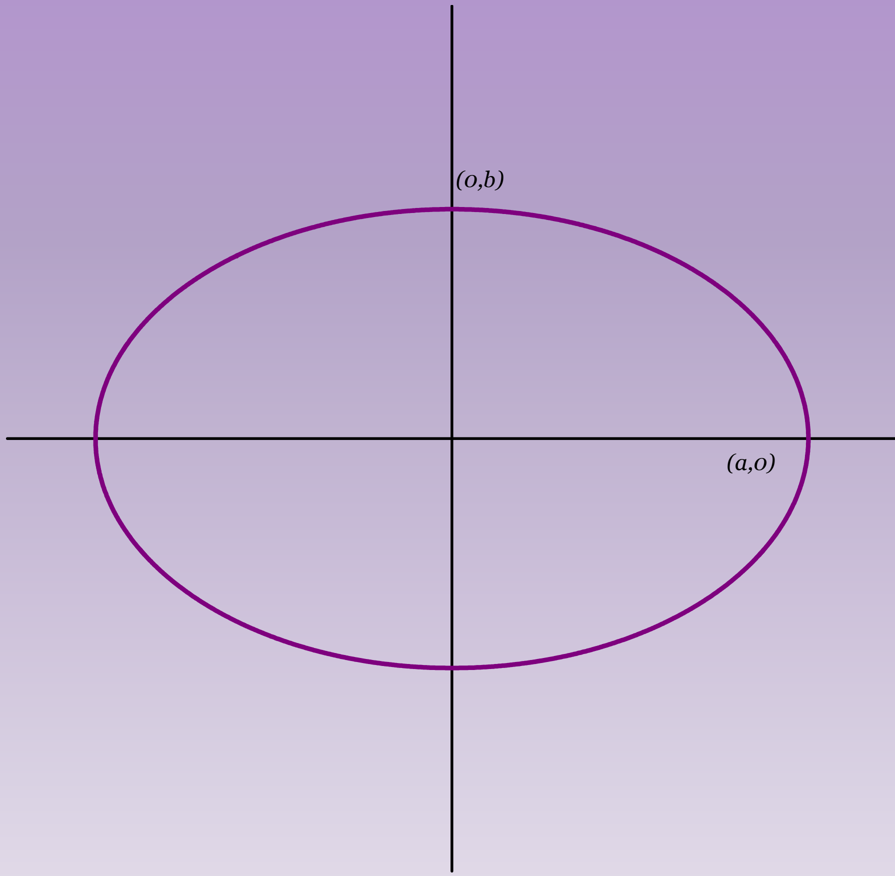
$$[s(x)]^2 - [c(x)]^2 = 1$$

$$s(x+y) = s(x)c(y) + c(x)s(y)$$

$$c(x+y) = c(x)c(y) + s(x)s(y)$$

Elliptic Computations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

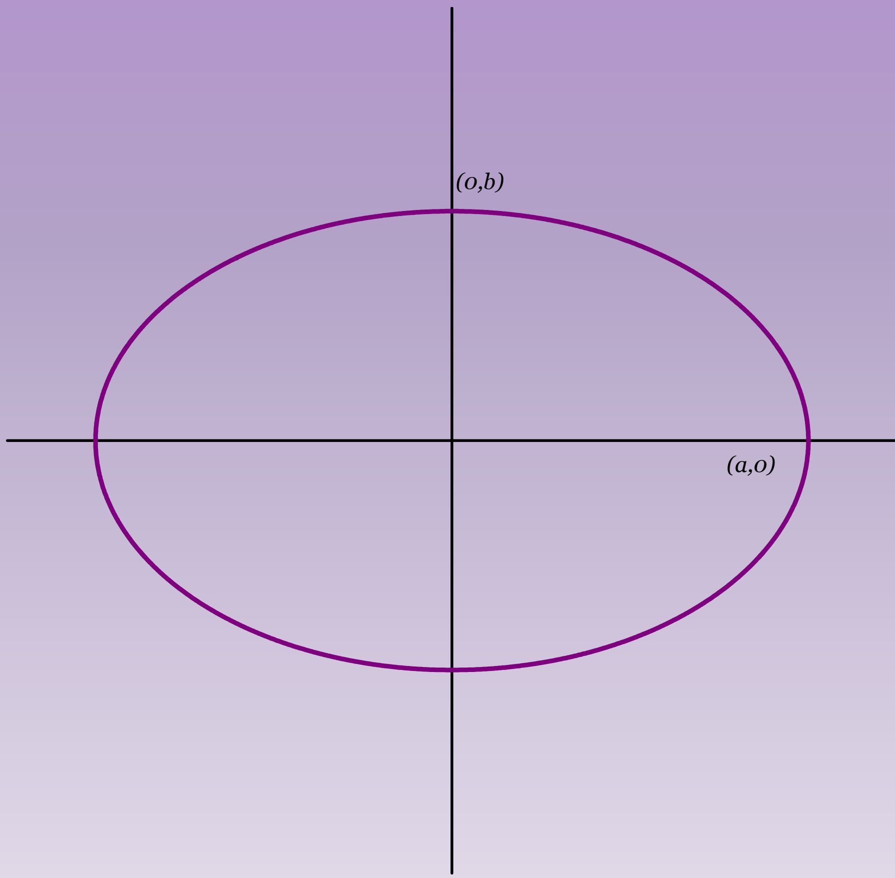


The area is πab – which can be computed by integrals.

What is the length of the curve???
Why did we care?
(Planetary orbits)

Elliptic Computations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



Parameterization:

Let $x = a \cos t$ and

$y = b \sin t$

This works well!!

Can we parameterize the parabola or the hyperbola similarly?

Not unless we use complex numbers!

Hyperbolic Functions

- Studied hyperbolic functions
- Used them to obtain solutions of cubics
- Found standard addition formulae for hyperbolic functions, derivatives and relation to the exponential function.

Vincenzo Riccati
1707-1775

Riccati's Work

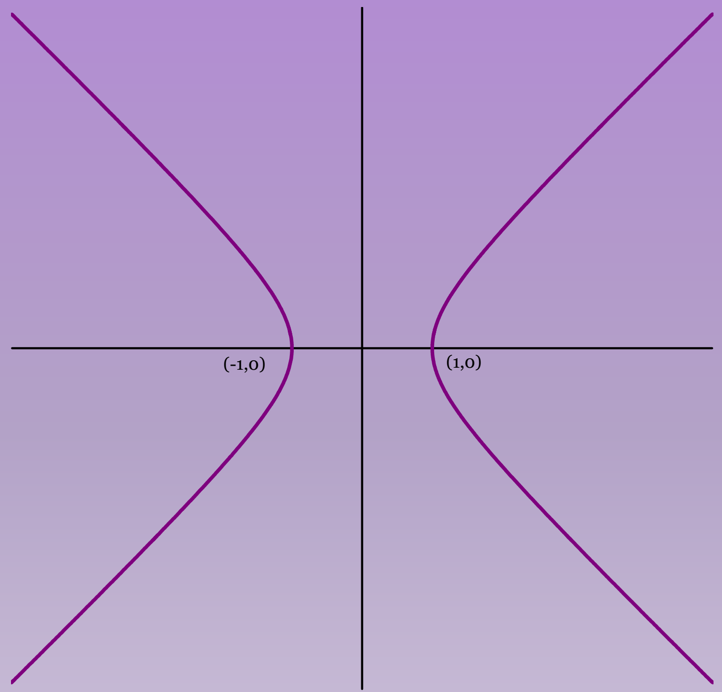
Developed the hyperbolic functions and proved their consistency using only geometry of the unit hyperbola $x^2 - y^2 = 1$ or $2xy = 1$.

Followed his father's interests in differential equations arising from geometrical problems.

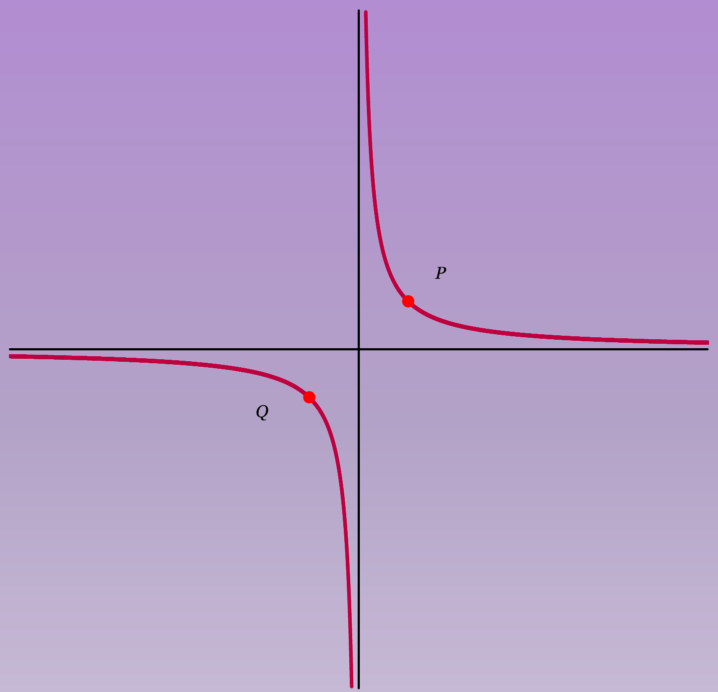
Led to study of the rectification of the conics in Cartesian coordinates and interest in the areas under the unit hyperbola.

Developed properties of the hyperbolic functions from purely geometrical considerations

Unit Hyperbolas



$$x^2 - y^2 = 1$$



$$2xy = 1$$

Circular Relation

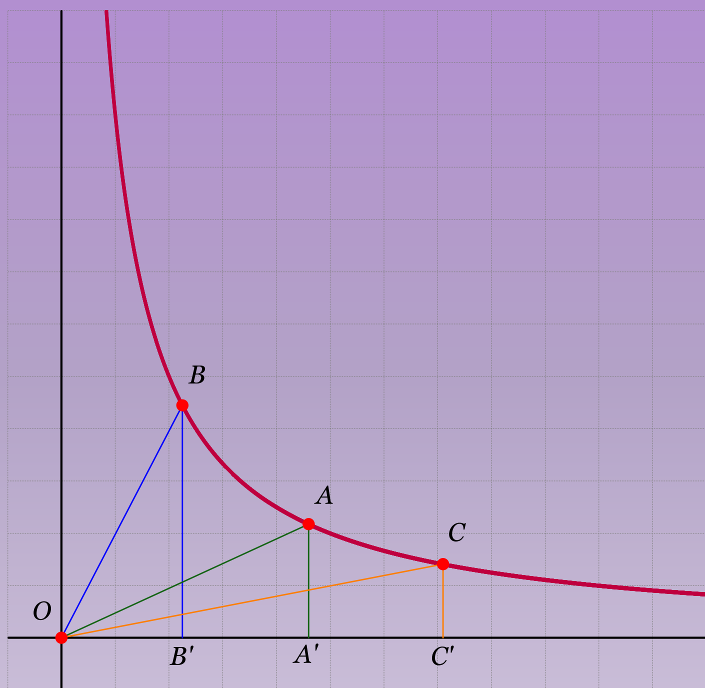
Arclength relation to area in a circle

$$\ell = r\theta \text{ and } A = r^2 \frac{\theta}{2} \Rightarrow \ell = \frac{2A}{r}$$

Does this hold in hyperbolas?

Not exactly!

Hyperbolic Areas



The graph is $xy = k$.

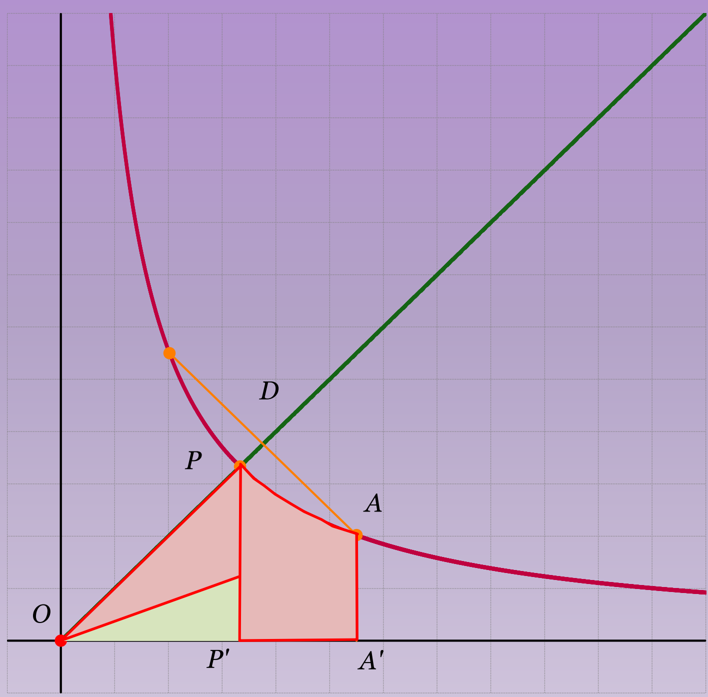
Area of $\Delta OAA'$

$$Area = \frac{xy}{2} = \frac{k}{2}$$

Note then that

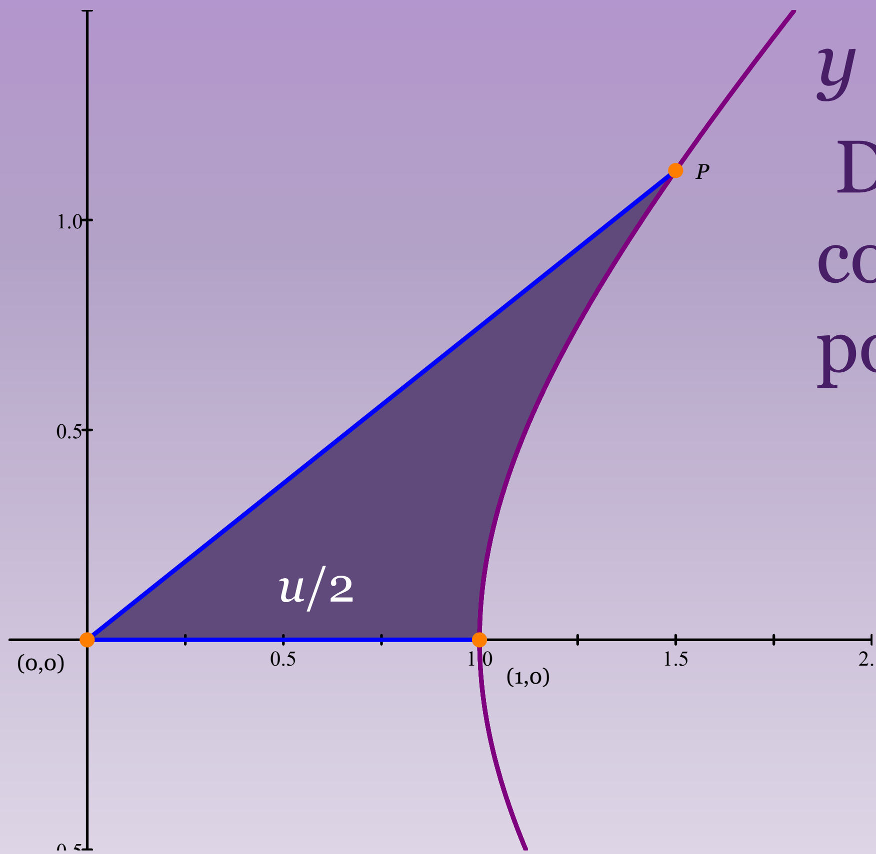
$$\text{Area}(\Delta OAA') = \text{Area}(\Delta OBB') = \text{Area}(\Delta OCC')$$

Area considerations for *xy*=1



1. Area($\Delta OAA'$)= Area($\Delta OAA'$)
2. Area(ΔOPQ) = Area(□AA′PP′)
   (Subtract Area($\Delta OQP'$) from above)
3. Area(APP′A′) = Area(OAP)
   (Add Area(QAP) to above)
4. $$Area(AOP) = \int_1^a \frac{dx}{x} = \ln(a) = u$$

Hyperbolic Trigonometric



Let $u/2$ = area bounded by x-axis, $y = x$, and curve.

Define the coordinates of the point P by

$x = \mathrm{ch}(u)$

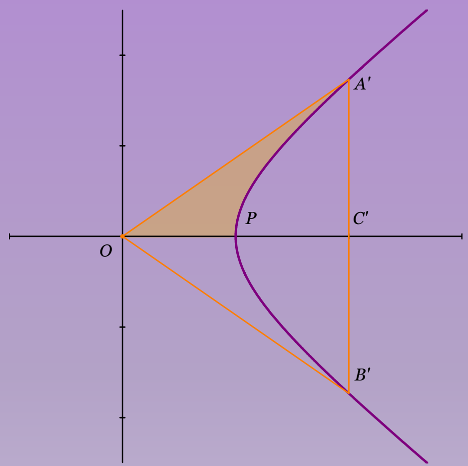
$y = \mathrm{sh}(u)$

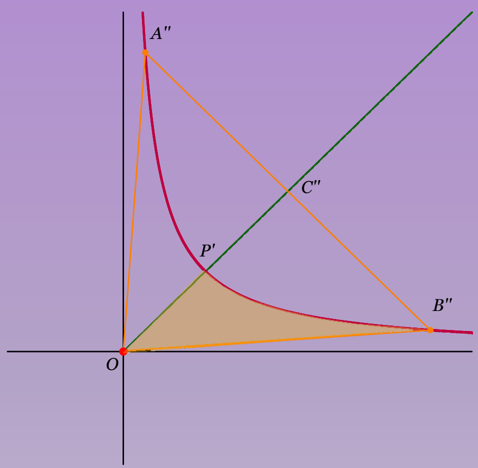
Properties of the functions ch(u) and sh(u)

1. $\mathrm{ch}(u)^2 - \mathrm{sh}(u)^2 = 1$ – This is immediately obvious.
2. $\mathrm{ch}(u + v) = \mathrm{ch}(u)\mathrm{ch}(v) + \mathrm{sh}(u)\mathrm{sh}(v)$
3. $\mathrm{sh}(u + v) = \mathrm{sh}(u)\mathrm{ch}(v) + \mathrm{ch}(u)\mathrm{sh}(v)$

We will prove 2 & 3 shortly.

Second Fundamental Property





Area(PC′A′)=Area(ΔOC′A′)-$u/2$

Rotate counterclockwise through $\pi/4$. We do not change area!!

Rotation carries $x^2 - y^2 = 1$ to $2xy = 1$

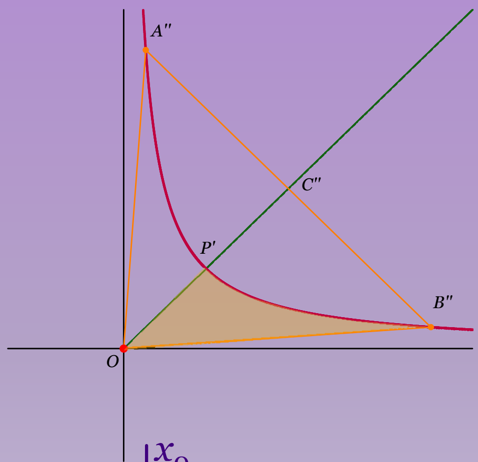
Second Fundamental Property

$B' = (\mathrm{ch}(u), -\mathrm{sh}(u)) = (c_1, s_1)$

$B'' = (x, 1/2x)$, $P' = (1/\sqrt{2}, 1/\sqrt{2})$

$B''C'' = B'C' = s_1$ and $OC'' = OC' = c_1$

Area bounded by OP', $y = 1/2x$ and OB':



$$K = \left[\frac{1}{2} \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}\right] + \int_{1/\sqrt{2}}^{x_0} \frac{dx}{2x} - \left[\frac{1}{2} \times x_0 \times \frac{1}{2x_0}\right] = \frac{1}{2}\ln x\Big|_{1/\sqrt{2}}^{x_0} = \frac{1}{2}\ln\left(\sqrt{2}x_0\right)$$

$$\frac{u}{2} = \frac{1}{2}\ln\left(\sqrt{2}x_0\right)$$

$$e^u = \sqrt{2}x_0 \Rightarrow x_0 = \frac{e^u}{\sqrt{2}} \quad \text{and} \quad y_0 = \frac{1}{\sqrt{2}e^u}$$

Second Fundamental Property

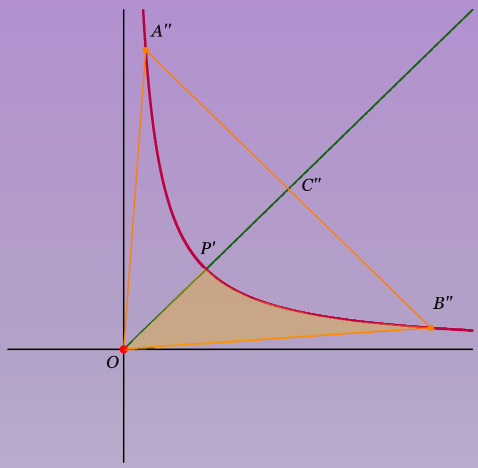
Now we need to find C''.

$B''C'' \perp OC''$ so it has slope -1 and equation

$$y - \frac{1}{2x_0} = -(x - x_0)$$

Since C'' lies on the diagonal, $x = y$ and

$$x = \frac{x_0}{2} + \frac{1}{4x_0} = y$$



Second Fundamental Property

Thus, the distance $B''C''$ is given by the distance formula

$$B''C'' = \left[\left(x_o - \left[\frac{x_o}{2} + \frac{1}{4x_o}\right]\right)^2 + \left(\frac{1}{2x_o} - \left[\frac{x_o}{2} + \frac{1}{4x_o}\right]\right)^2\right]^{1/2}$$

$$= \sqrt{2}\left(\frac{x_o}{2} - \frac{1}{4x_o}\right)$$

This last term is positive if $x_o > 1/\sqrt{2}$

Second Fundamental Property

Recall that $x_o = e^u/\sqrt{2}$ and $B''C'' = \text{sh}(u)$. Therefore

$$\text{sh}(u) = \sqrt{2}\left(\frac{e^u}{2\sqrt{2}} - \frac{\sqrt{2}}{4e^u}\right) = \frac{e^u - e^{-u}}{2}$$

Second Fundamental Property

$$O''C'' = \left[\left(\left[\frac{x_0}{2} + \frac{1}{4x_0} \right] \right)^2 + \left(\left[\frac{x_0}{2} + \frac{1}{4x_0} \right] \right)^2 \right]^{1/2}$$

$$= \sqrt{2} \left(\frac{x_0}{2} + \frac{1}{4x_0} \right)$$

$$\mathrm{ch}(u) = \sqrt{2} \left(\frac{e^u}{2\sqrt{2}} + \frac{\sqrt{2}}{4e^u} \right) = \frac{e^u + e^{-u}}{2}$$

Hyperbolic Trigonometric Functions

Traditionally, we have:

$$\text{ch}(u) = \cosh(u)$$

$$\text{sh}(u) = \sinh(u)$$

Define the remaining 4 hyperbolic trig functions as expected:

$$\tanh(u), \coth(u), \text{sech}(u), \text{csch}(u)$$

Hyperbolic Trig Functions

The hyperbolic trigonometric functions satisfy the following properties

$$\cosh(x)^2 - \sinh(x)^2 = 1$$

$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$$

$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$$

$$\cosh(-x) = \cosh(x)$$

$$\sinh(-x) = -\sinh(x)$$

Hyperbolic Trig Identities

$$\cosh^2 x - \sinh^2 x = 1$$

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$\tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$

$$\sinh^2 \frac{x}{2} = \frac{\cosh x - 1}{2} \text{ and } \cosh^2 \frac{x}{2} = \frac{\cosh x + 1}{2}$$

Hyperbolic Trig Functions

Since the exponential function has a power series expansion

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

The hyperbolic trig functions have power series expansions

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \text{ and } \sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

Hyperbolic Trig Functions

Recall that the Maclaurin series for the sine and cosine are:

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \text{ and } \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

VERY SIMILAR!!!!

Hyperbolic Trig Functions

Replace x by ix, where $i^2 = -1$.:

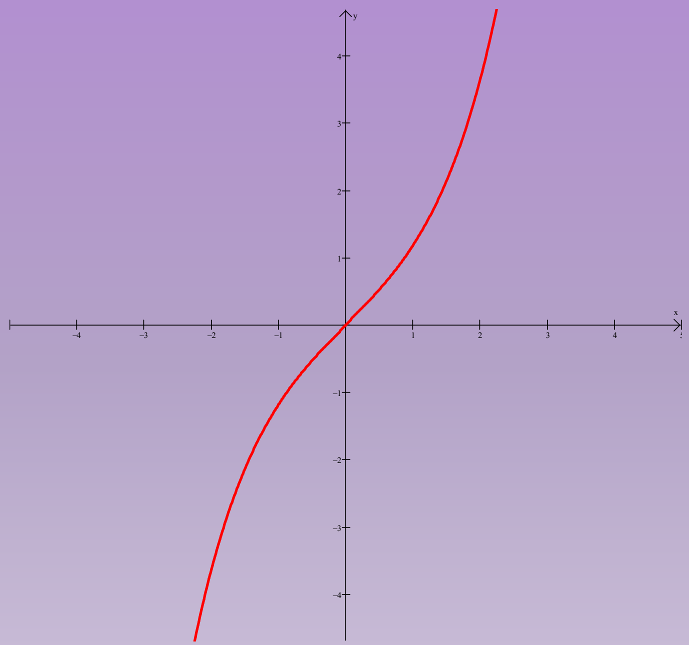
$$\cos(ix) = \sum_{n=0}^{\infty} \frac{(-1)^n (ix)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n i^{2n} x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = \cosh x$$

$$\sin(ix) = \sum_{n=0}^{\infty} \frac{(-1)^n (ix)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n i^{2n+1} x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{ix^{2n+1}}{(2n+1)!} = i \sinh x$$

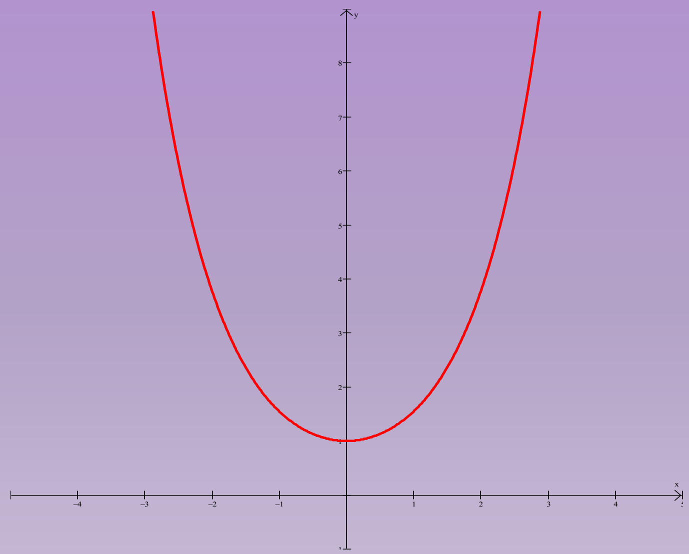
$$\cosh x = \cos(ix) = \cos\left(\frac{x}{i}\right)$$

$$\sinh x = -i \sin(ix) = i \sin\left(\frac{x}{i}\right)$$

The Graphs



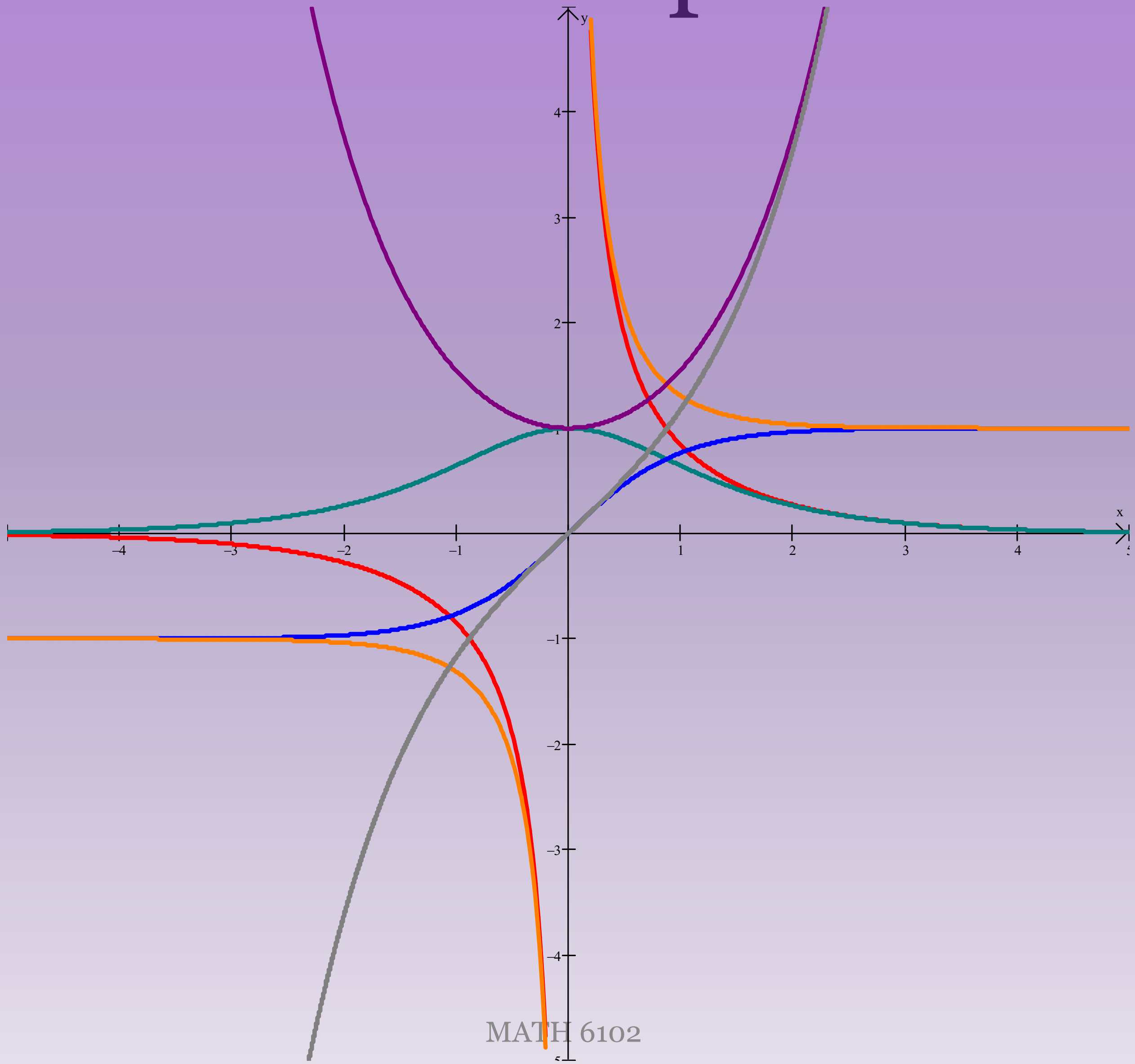
$y = \sinh(x)$



$y = \cosh(x)$

The catenary

The Graphs



The Derivatives

$$\frac{d}{dx}\sinh x = \cosh x \qquad \frac{d}{dx}\cosh x = \sinh x$$

The Derivatives

$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x$$

$$\frac{d}{dx}\coth x = -\operatorname{csch}^2 x$$

$$\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}\operatorname{csch} x = -\operatorname{csch} x \coth x$$

The Inverse Functions

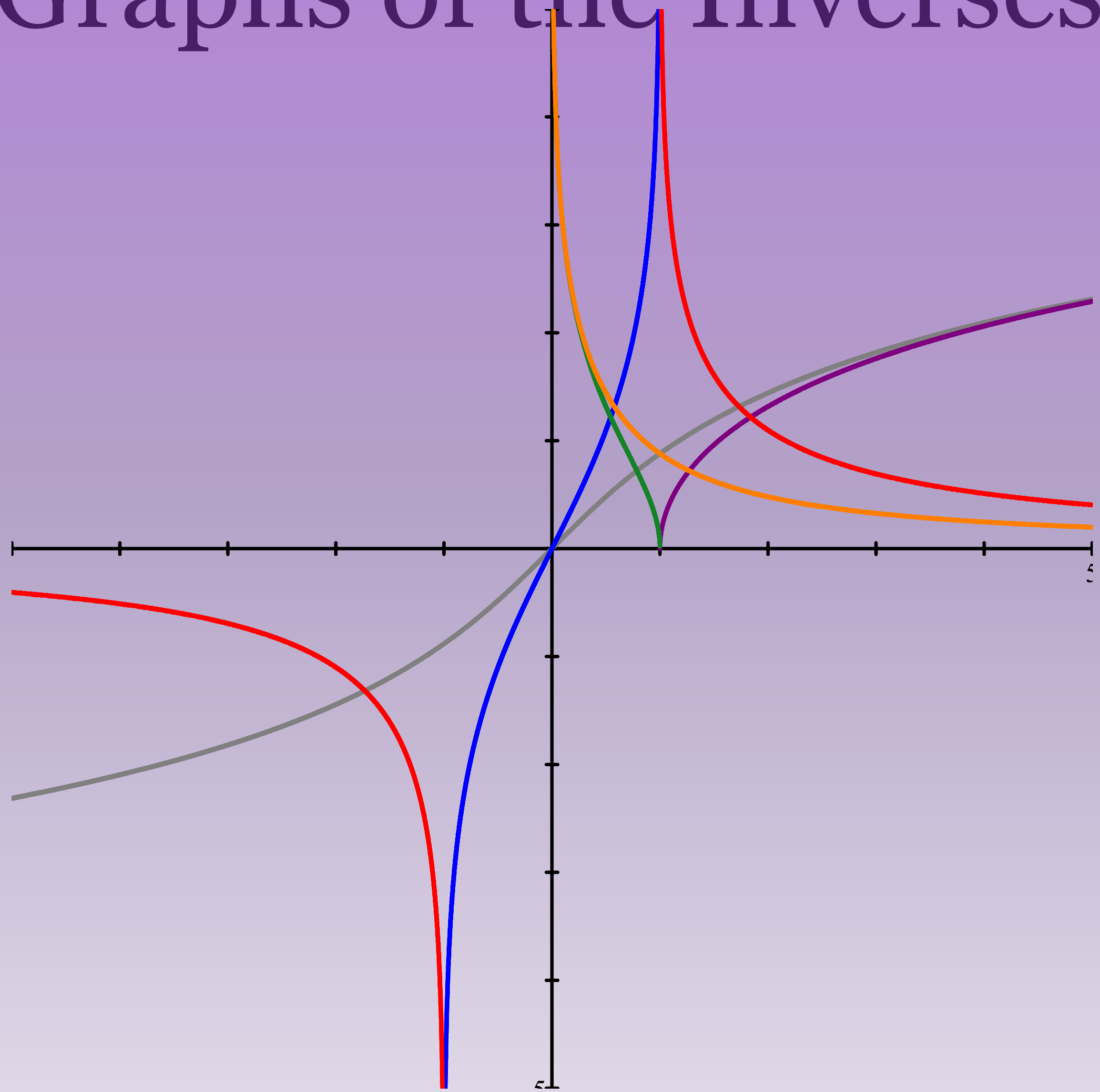
$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right) \quad \cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)$$

$$\tanh^{-1} x = \ln\left(\sqrt{\frac{1+x}{1-x}}\right) \quad \coth^{-1} x = \ln\left(\sqrt{\frac{x+1}{x-1}}\right)$$

$$\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$$

$$\operatorname{csch}^{-1} x = \ln\left(\frac{1+\sqrt{1+x^2}}{x}\right)$$

Graphs of the Inverses



The Lambert W Function

This is also known as the omega function (ω) or the `ProductLog` function (in *Mathematica*). In *Maple* it is known as the Lambert W function.

Arose from Lambert's work to solve:

$$x = q + x^m$$

Euler extended it to

$$x^\alpha - x^\beta = (\alpha - \beta)\upsilon x^{\alpha+\beta}$$

The Lambert W Function

From

$$x = q + x^m$$

Substitute $x^{-\beta}$ for x, set $m = \alpha/\beta$ and $q = (\alpha - \beta)\upsilon$

$$x = q + x^m$$

$$x^{-\beta} = (\alpha - \beta)\upsilon + x^{-\alpha}$$

$$x^{-\beta} - x^{-\alpha} = (\alpha - \beta)\upsilon$$

$$\frac{1}{x^{\beta}} - \frac{1}{x^{\alpha}} = (\alpha - \beta)\upsilon$$

$$x^{\alpha} - x^{\beta} = (\alpha - \beta)\upsilon x^{\alpha+\beta}$$

The Lambert W Function

Divide by α - β and let α go to β.

$$\lim_{\beta \to \alpha} \frac{x^{\alpha} - x^{\beta}}{\alpha - \beta} = \lim_{\beta \to \alpha} \upsilon x^{\alpha+\beta}$$

$$\frac{d}{d\alpha} x^{\alpha} = \upsilon x^{2\alpha}$$

$$x^{\alpha} \log x = \upsilon x^{2\alpha}$$

$$\log x = \upsilon x^{\alpha}$$

The Lambert W Function

If we can solve this for $\alpha = 1$, then we can solve it for any α.

$$\log x = v x^{\alpha}$$

$$\alpha \log x = \alpha v x^{\alpha}$$

$$\log x^{\alpha} = \alpha v x^{\alpha}$$

$$\text{Set } z = x^{\alpha} \text{ and } u = \alpha v$$

$$\log z = u z$$

Then apply Euler's solution to get:

$$\log x = v + \frac{2^1}{2!} v^2 + \frac{3^2}{3!} v^3 + \frac{4^3}{4!} v^4 + \frac{5^4}{5!} v^5 + \cdots$$

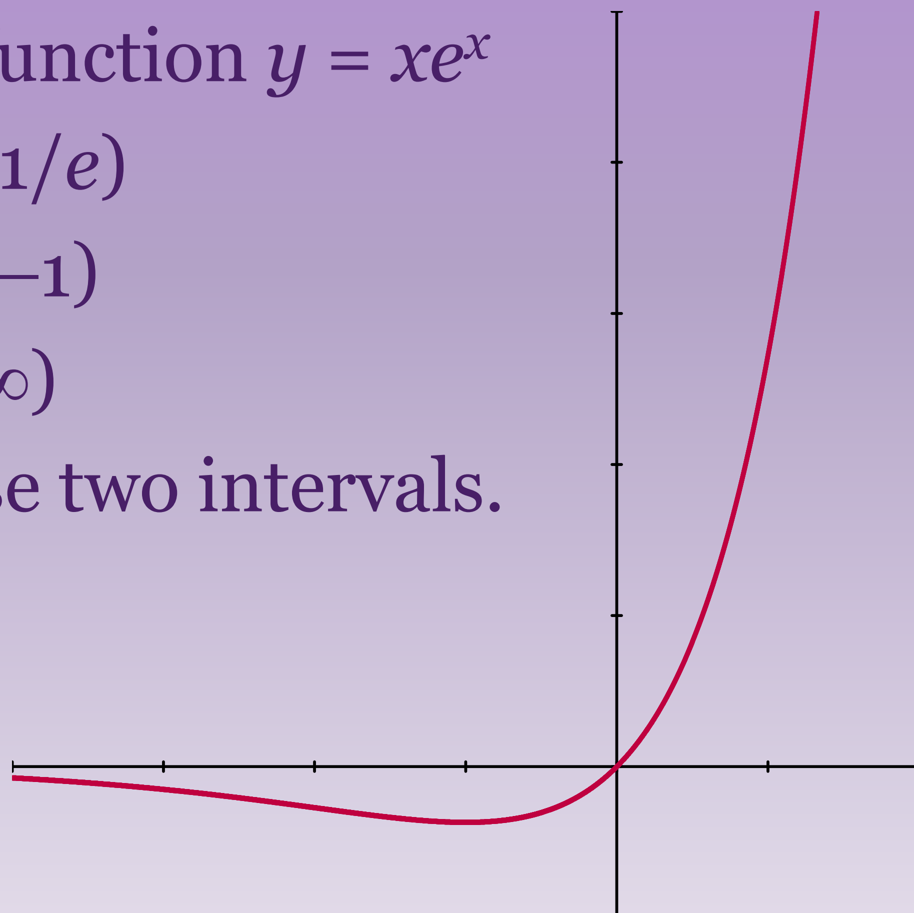
The Lambert W Function

From the work of Wright *et al* after 1925, more is known.

Consider the product function $y = xe^x$

- Local min: $(-1, -1/e)$
- Decreasing: $(-\infty,-1)$
- Increasing: $(-1,\infty)$

Has an inverse on those two intervals.

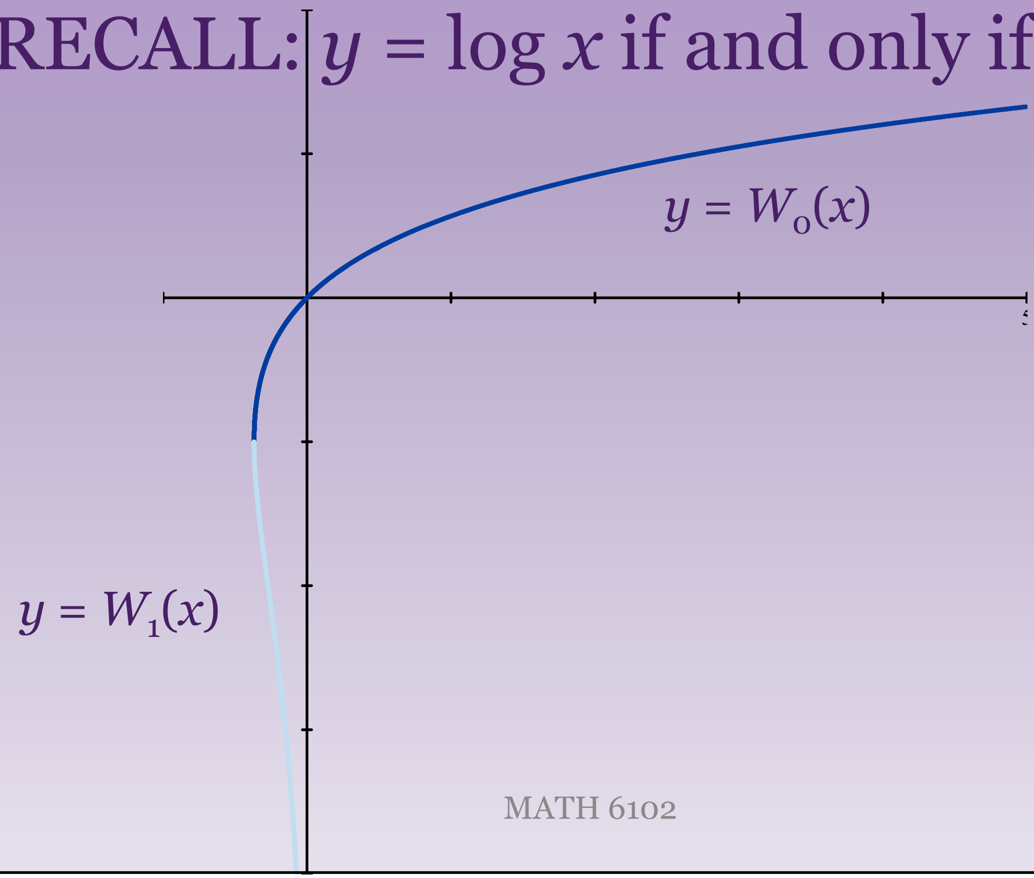


The Lambert W Function

For $x > -1$, we define the Lambert W function $W(x)$ by.

$$y = W(x) \text{ if and only if } x = ye^y$$

[RECALL: $y = \log x$ if and only if $x = e^y$]



The Lambert W Function

$W(x)$ solves a broader class of equations:

1. Solve $y = xe^{-x}$ for x.

   The graph is the graph of $y = xe^x$ flipped across both the x- and y-axes. Thus, we should have that $x = -W(-y)$.

   Check:

$$x = -W(-y) \text{ if and only if } -x = W(-y)$$
$$\text{if and only if } -y = -xe^{-x}$$
$$\text{if and only if } y = xe^{-x}$$

The Lambert W Function

$W(x)$ solves a broader class of equations:

2. Solve $a = xb^x$ for x. Rewrite this to the form $z = ue^u$ whose solution is $u = W(z)$

$$a = xb^x$$

$$a = xe^{x\ln b}$$

$$a\ln b = x\ln b e^{x\ln b}$$

$$\text{Set } z = a\ln b \text{ and } u = x\ln b$$

$$u = W(z)$$

$$x\ln b = W(a\ln b)$$

$$x = W(a\ln b)/\ln b$$

The Lambert W Function

3. Solve $b = x^{x^a}$ for x. Rewrite this to the form $z = ue^u$ whose solution is $u = W(z)$

$$b = x^{x^a}$$

$$b = e^{x^a \ln x} = e^{(\ln x)e^{a\ln x}}$$

$$b^a = e^{(a\ln x)e^{a\ln x}}$$

$$e^{a\ln b} = e^{(a\ln x)e^{a\ln x}}$$

$$a\ln b = (a\ln x)e^{a\ln x}$$

Set $z = a\ln b$ and $u = a\ln x$

$$u = W(z)$$

$$a\ln x = W(a\ln b)$$

$$x = e^{W(a\ln b)/a}$$

The Lambert W Function

4. Solve $x + b = a^x$ for x.

$$x+b=a^x. \text{ Put } u=x+b$$

$$u=a^{u-b}=a^{-b}a^u$$

$$a^{-b}=ua^{-u}$$

$$-a^{-b}=-ua^{-u}$$

$$-a^{-b}=va^v, \text{ where } v=-u$$

By 2

$$v=W(-a^{-b}\ln a)/\ln a$$

$$x=-b-W(-a^{-b}\ln a)/\ln a$$

The Lambert W Function

5. Consider $h(x) = x^{x^{x^{x^{\cdots}}}}$ - an infinite exponential tower. When does it converge? Definitely at $x = 1$ and for $0 < x < 1$. Anywhere else? Euler showed that it converged for $e^{-e} < x < e^{1/e}$. Wright showed that when it converges it converges to

$$x^{x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}} = \frac{W(-\ln x)}{-\ln x}$$

The Lambert W Function
Its Derivative

$$x = W(x)e^{W(x)}$$

$$\frac{dx}{dx} = W'(x)e^{W(x)} + W(x)W'(x)e^{W(x)}$$

$$1 = W'(x)\left[e^{W(x)} + W(x)e^{W(x)}\right]$$

$$W'(x) = \frac{1}{e^{W(x)} + W(x)e^{W(x)}} = \frac{1}{e^{W(x)}\left(1+W(x)\right)}$$

$$= \frac{1}{\frac{x}{W(x)}\left(1+W(x)\right)}$$

$$W'(x) = \frac{W(x)}{x\left(1+W(x)\right)}; \quad x \neq 0, -1/e$$

The Lambert W Function
Its Maclaurin Series

We find that the power series expansion is

$$W(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n$$

Not from first principles, rather from series reversion of

$$ze^z = \sum_{n=0}^{\infty} \frac{z^{n+1}}{n!}$$

The Lambert W Function
Its Antiderivative

Using Integration by Parts it can be shown that

$$\int W(x)\,dx = \frac{x\left(W^2(x) - W(x) + 1\right)}{W(x)} + C$$

The Lambert W Function
Some of its Values

$$W(e) = 1$$

$$W(0) = 0$$

$$W\left(-\frac{1}{e}\right) = -1$$

$$W\left(-\frac{\ln 2}{2}\right) = -\ln 2$$

$$W(1) = \Omega = 0.567143290409783872999686622$$

$$e^{-\Omega} = \Omega \text{ or } \ln\left(\frac{1}{\Omega}\right) = \Omega$$

$$W\left(-\frac{\pi}{2}\right) = \frac{i\pi}{2}$$