MATH 6102
Spring 2009

The Derivative

History

- Tangent line — familiar to Greek geometers such as Euclid (c. 300 BCE), Archimedes (c. 287–212 BCE) and Apollonius of Perga (c. 262–190 BCE).
- Use of infinitesimals — found in Indian mathematics, (as early as 500?) — Aryabhata (476–550) used infinitesimals to study the motion of the moon.
- Use of infinitesimals to compute rates of change — developed by Bhāskara II (1114-1185)
- Bhaskara — it is said that many key notions of differential calculus are found in his work, such as *Rolle's theorem*.

History

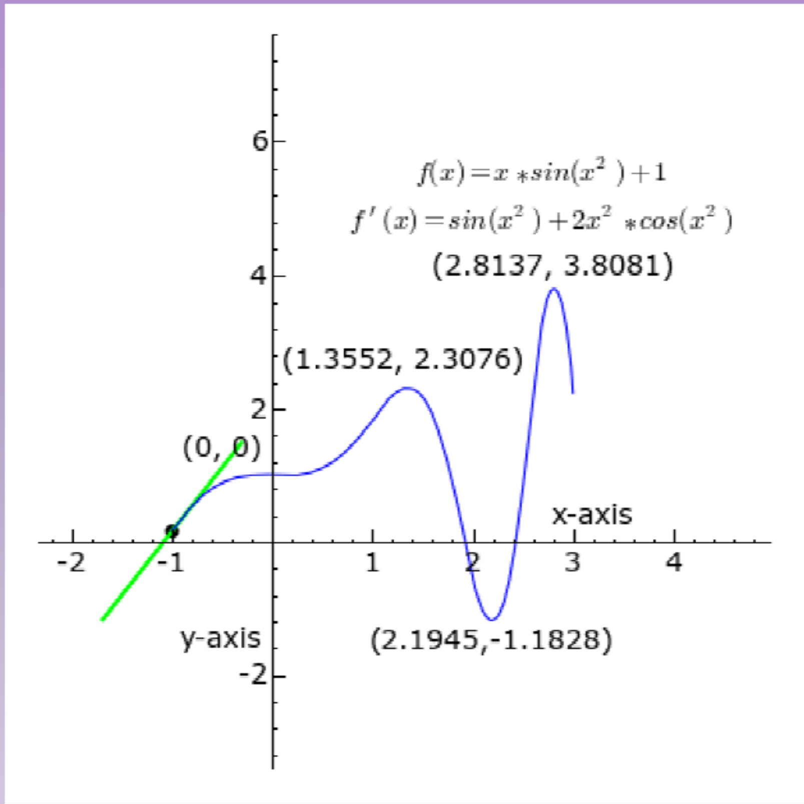
- Sharaf al-Dīn al-Tūsī (1135-1213) — Persia —first to discover derivative of cubic polynomials
- *Treatise on Equations* — developed derivative function and maxima and minima of curves — to solve cubic equations which may not have positive solutions.
- Early version of *Mean Value Theorem* first described by Parameshvara (1370–1460) from the Kerala school in his commentary on Bhaskara II.
- Early work by Isaac Barrow (1630 – 1677), René Descartes (1596 – 1650), Christiaan Huygens (1629 – 1695), Blaise Pascal (1623 – 1662) and John Wallis (1616 – 1703).
- Isaac Barrow is often credited with the early development of the derivative.
- Newton and Leibniz : Newton applied differentiation to theoretical physics; Leibniz developed much of the notation still used today.

Tangent Line to a Curve

The question of finding the tangent line to a graph, or the **tangent line problem**, was one of the central questions leading to the development of calculus in the 17th century. In the second book of his *Geometry*, René Descartes said of the problem of constructing the tangent to a curve,

"*And I dare say that this is not only the most useful and most general problem in geometry that I know, but even that I have ever desired to know.*"

Tangent Line



http://mathworld.wolfram.com/TangentLine.html

Definition

The Difference Quotient

$$\frac{f(x)-f(a)}{x-a}$$

or, by letting $x = a + h$

$$\frac{f(a+h)-f(a)}{h}$$

These measure the slope of the secant line.

Definition

The tangent line is defined to be the limit of the secant line as *x* gets close to *a*, so

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

or

$$m = f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

These measure the slope of the tangent line.

Definition

Symmetric Difference Quotient

$$m = f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a-h)}{2h}$$

This limit tends to converge faster than the regular difference quotient, BUT it is possible for this limit to exist and the function not have a derivative at a!

Example

Find the slope of the tangent line to $y = 3x^2 - 4x + 1$ at $x = 1$.

<u>Difference Quotient</u>:

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h\to 0} \frac{\left(3(1+h)^2 - 4(1+h) + 1\right) - 0}{h}$$

$$= \lim_{h\to 0} \frac{6h + 3h^2 - 4h}{h} = \lim_{h\to 0}(2 + 3h) = 2$$

Example

Method 2:

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x \to 1} \frac{\left(3x^2 - 4x + 1\right) - 0}{x - 1}$$

$$= \lim_{x \to 1} \frac{(3x - 1)(x - 1)}{x - 1} = \lim_{x \to 1} (3x - 1) = 2$$

Example

Method 3:

$$f'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1-h)}{2h}$$

$$=\lim_{h\to 0}\frac{\left(3(1+h)^2-4(1+h)+1\right)-\left(3(1-h)^2-4(1-h)+1\right)}{2}$$

$$=\lim_{h\to 0}\frac{(3+6h+3h^2-4-4h+1)-(3-6h+3h^2-4+4h+1)}{2h}$$

$$=\lim_{h\to 0}\frac{4h}{2h}=2$$

Example

Which method was better?

Why?

Will this always be the case?

Example

Find the slope of the tangent line to $y = \sin(x)$ at $x = 0$.

Difference Quotient:

$$f'(1) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(h)}{h} = ?$$

Example

Methods 2 & 3:

$$f'(1) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} = ?$$

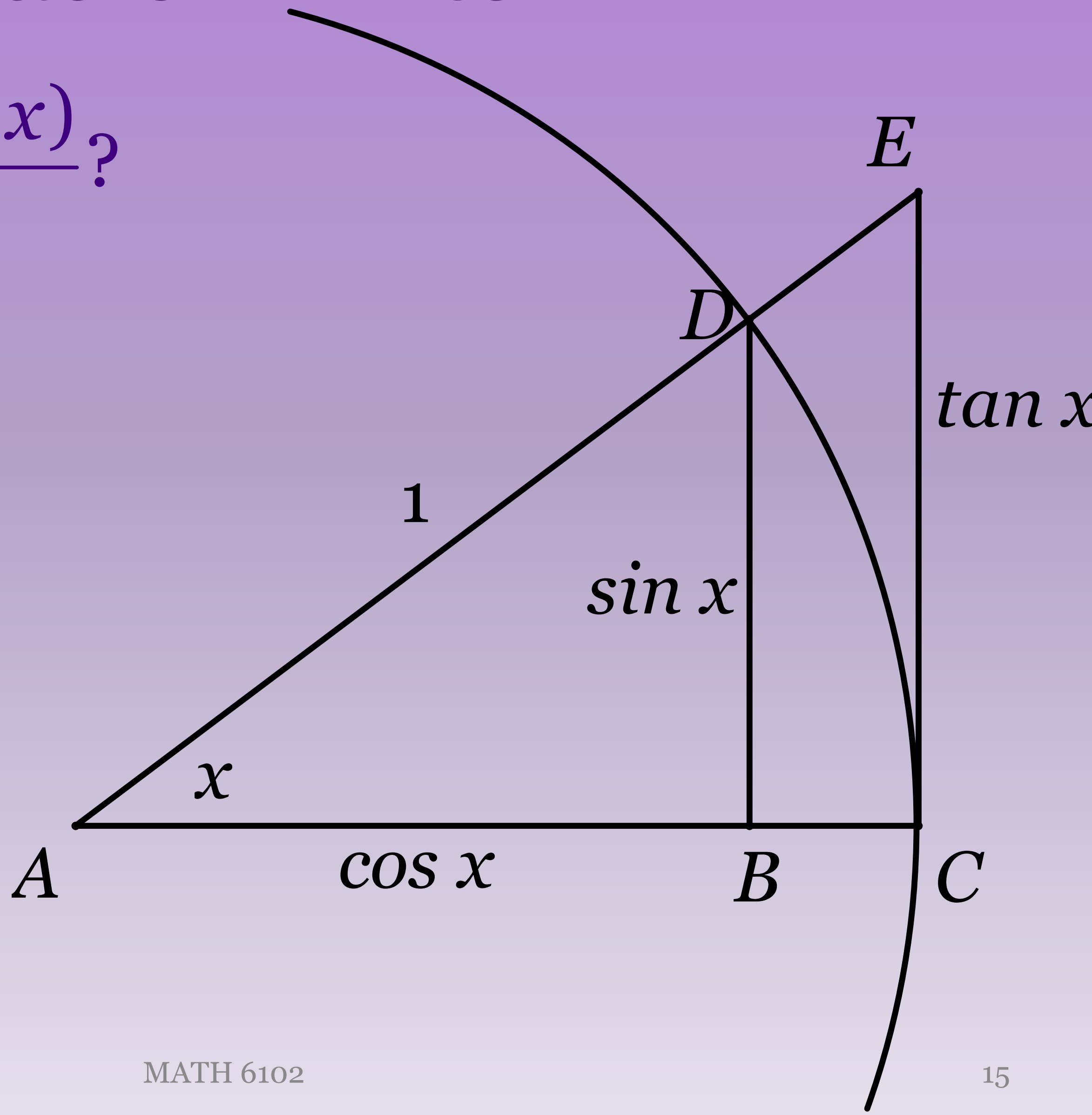
$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0-h)}{2h}$$

$$= \lim_{h \to 0} \frac{\sin(h) - \sin(-h)}{2h}$$

$$= \lim_{h \to 0} \frac{2\sin(h)}{2h} = ?$$

Basic Limits

What is $\lim_{x \to 0} \frac{\sin(x)}{x}$?



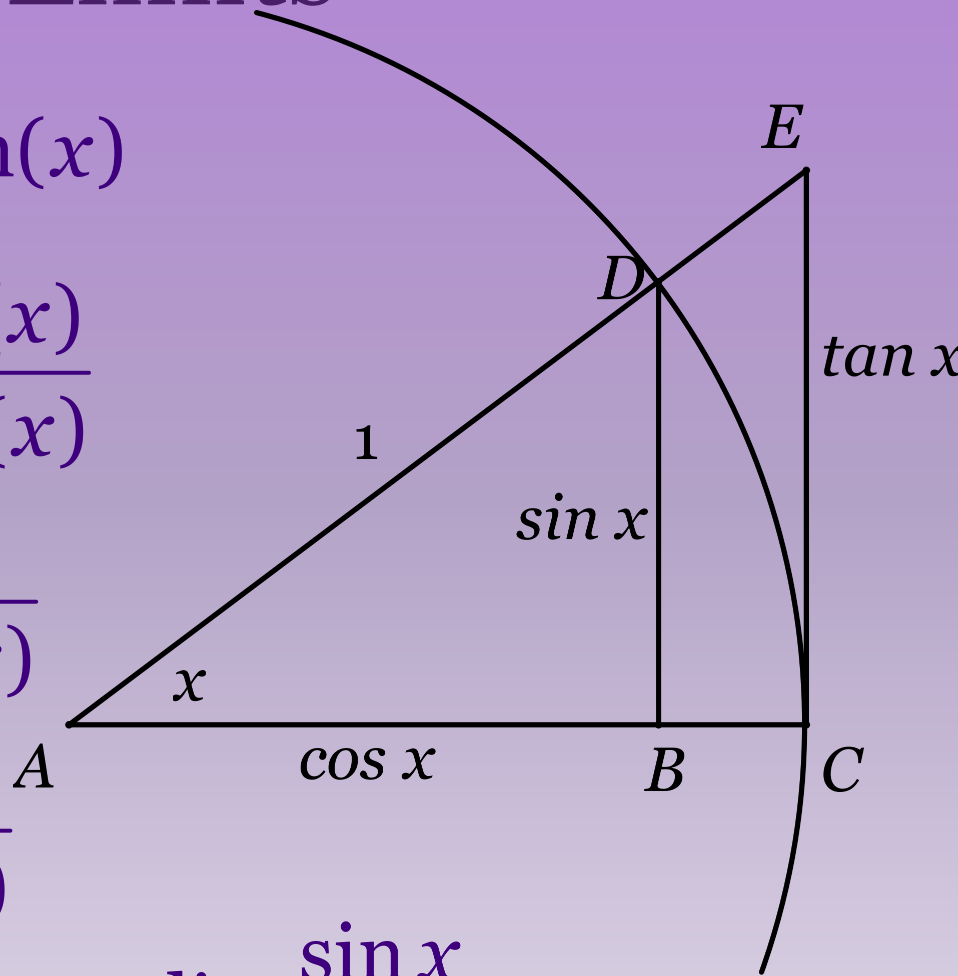
Basic Limits

$$\frac{1}{2}\sin(x)\cos(x) < \frac{x}{2} < \frac{1}{2}\tan(x)$$

$$\sin(x)\cos(x) < x < \frac{1}{2}\frac{\sin(x)}{\cos(x)}$$

$$\cos(x) < \frac{x}{\sin(x)} < \frac{1}{\cos(x)}$$

$$\cos(x) < \frac{\sin x}{x} < \frac{1}{\cos(x)}$$



$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

Basic Definition

Let f be defined on an open interval (a,b). If $x \in (a,b)$, then we say that f is *differentiable* at x if

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

and this limit exists.

If f is differentiable at all $x \in (a,b)$, we say that f is differentiable <u>on the interval</u>.

Alternate Definition

If f is defined at a in some open interval I then

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

if these limits exist. Also, if these limits exist, then

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a-h)}{2h}$$

$f'(a)$ is called the *derivative* of f at a.

Operator

The derivative is a operator that maps a function to its derived function. Sometimes, the notation used is D_x.

Various notations

$$f'(x) = \frac{d}{dx} f(x) = D_x f(x) = \dot{f}(x)$$

Leibniz notation

Newton notation

Properties

$$(f+g)'(x) = f'(x) + g'(x)$$

$$(kf)'(x) = kf'(x), \quad k \text{ constant}$$

$$(f-g)'(x) = f'(x) - g'(x)$$

Product Rule

$$(f \times g)'(x) = f(x)g'(x) + f'(x)g(x) +$$

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + \frac{df}{dx}g$$

Notation

An operator that satisfies

$$\Phi(f+g)=\Phi(f)+\Phi(g)$$

and

$$\Phi(kf)=k\Phi(f), k \text{ constant}$$

is called a *linear* operator.

An operator that satisfies

$$\Phi(fg)=f\Phi(g)+\Phi(f)g$$

is called a *derivation*.

Quotient Rule

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) + f(x)g'(x) +}{(g(x))^2}$$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} + f\frac{dg}{dx}}{g^2}$$

Application

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{\cos x \frac{d}{dx}\sin x - \sin x \frac{d}{dx}\cos x}{\cos^2 x}$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

Chain Rule

$$\frac{d}{dx}\left(f \circ g\right)(x) = \frac{df}{dx}\left(g(x)\right)\frac{d}{dx}g(x)$$
$$= f'(g(x)) \bullet g'(x)$$

Possibly the most misunderstood and maligned derivative rule!!

Inverse Functions

Assume that f is one-to-one and onto and it has an inverse function, g. What is the graph of $g(x)$ compared to the graph of $f(x)$?

What is $f'(x)$? What does it measure?

What should happen to the slope of a line when reflected across the line $y = x$?

Inverse Functions

$$x = f(g(x))$$

$$\frac{d}{dx}x = f'(g(x))g'(x)$$

$$1 = f'(g(x))g'(x)$$

$$g'(x) = \frac{1}{f'(g(x))}$$

So if $f(a) = b$, then $g(b) = a$ and

$$g'(b) = \frac{1}{f'(a)}$$

Application

Find the derivative of the arccosine function.

$$x = \cos(\arccos(x))$$

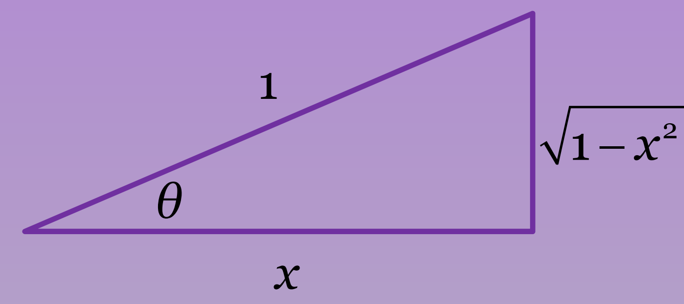
$$1 = -\sin(\arccos(x))\frac{d}{dx}(\arccos x)$$

$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sin(\arccos(x))}$$

Application

This is not really satisfactory!!

$\theta = \arccos(x) \Leftrightarrow x = \cos(\theta)$



$$\sin\theta = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2} \Rightarrow \sin(\arccos(x)) = \sqrt{1-x^2}$$

Thus

$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sin(\arccos(x))}$$

$$= -\frac{1}{\sqrt{1-x^2}}$$

Exponential Functions

This is based off the following limit:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e=\lim_{t\to 0}\left(1+t\right)^{1/t}$$

Thus $\displaystyle \frac{d}{dx}\ln x=\lim_{h\to 0}\frac{\ln(x+h)-\ln(x)}{h}=\lim_{h\to 0}\frac{1}{h}\ln\left(\frac{x+h}{x}\right)$

$$=\lim_{h\to 0}\frac{1}{h}\ln\left(1+\frac{h}{x}\right)=\lim_{h\to 0}\frac{1}{x}\frac{x}{h}\ln\left(1+\frac{h}{x}\right)$$

$$=\frac{1}{x}\lim_{h\to 0}\ln\left(1+\frac{h}{x}\right)^{\frac{x}{h}}\overset{?}{=}\frac{1}{x}\ln\left(\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{\frac{x}{h}}\right)$$

$$=\frac{1}{x}\ln\left(\lim_{t\to 0}\left(1+t\right)^{1/t}\right)=\frac{1}{x}\ln e=\frac{1}{x}$$

Exponential Functions

Using the Inverse Function Theorem we have:

$$\frac{d}{dx}e^x = e^x$$

Since all logs and exponentials are just multiples of the natural log and exponential:

$$a^x = e^{x \ln a} \quad \Rightarrow \quad \frac{d}{dx}a^x = a^x \ln a$$

$$\log_b x = \frac{\ln x}{\ln b} \quad \Rightarrow \quad \frac{d}{dx}\log_b x = \frac{1}{x \ln b}$$

Fermat's Theorem

Let f be defined on an open interval containing c. If f assumes its maximum or minimum value at $x = c$, and if f is differentiable at $x = c$, then

$$f'(c) = 0.$$

Rolle's Theorem

Let f be continuous on [a,b] and be differentiable on (a,b) and satisfy f (a) = f (b). Then there exists at least one $c \in (a,b)$ *so that* $f'(c) = 0.$

Mean Value Theorem

Let f be a continuous function on [a,b] that is differentiable on (a,b). Then there is a point $c \in (a,b)$ *such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Corollary 1

Let f be a differentiable function on (a,b) such that $f'(x)=0$ for all $x \in (a,b)$. Then f is a constant function on (a,b).

$$0 = f'(c) = \frac{f(b) - f(a)}{b - a} \Rightarrow f(b) = f(a)$$

Corollary 2

Let f and g be a differentiable functions on (a,b) such that $f' = g'$ *for all* $x \in (a,b)$. *Then there is a constant C so that* $f(x) = g(x) + C$ *on (a,b).*

Corollary 2

Let f and g be a differentiable functions on (a,b) such that $f' = g'$ for all $x \in (a,b)$. Then there is a constant C so that $f(x) = g(x) + C$ on (a,b).

L'Hospital's Rule

Let M signify a, a^+, a^-, ∞, or $-\infty$, where a is a real number, and suppose that f and g are differentiable functions for which the following limit exists:

$$\lim_{x\to M}\frac{f'(x)}{g'(x)}=L$$

If

$$\lim_{x\to M} f'(x)=\lim_{x\to M} g'(x)=0$$

or if

$$\lim_{x\to M}\left|g'(x)\right|=+\infty$$

then

$$\lim_{x\to M}\frac{f(x)}{g(x)}=L$$

L'Hospital's Rule

$$\lim_{x \to 0^+} x \log x = \lim_{x \to 0^+} \frac{\log x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} -x = 0$$

$$\lim_{x \to 0^+} x^x = 0^0 = ?$$

$$\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} e^{x \log x} = e^0 = 1$$