

Integral Calculus

Introduction to Integration

Integral calculus is the study of the definitions, properties, and applications of two related concepts, the *indefinite integral* and the *definite integral*. The process of finding the value of an integral is called *integration*. In technical language, integral calculus studies two related linear operators.

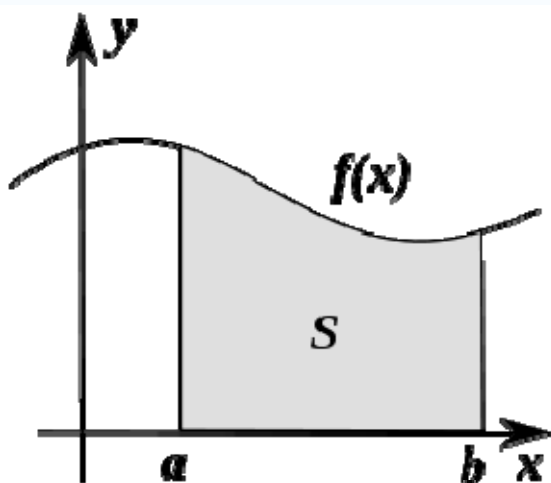
The *indefinite integral* is the *antiderivative*, the inverse operation to the derivative. F is an indefinite integral of f when f is a derivative of F . (This use of upper- and lower-case letters for a function and its indefinite integral is common in calculus.)

The *definite integral* inputs a function and outputs a number, which gives the area between the graph of the input and the x -axis. The technical definition of the definite integral is the limit of a sum of areas of rectangles, called a Riemann sum.

A motivating example is the distances traveled in a given time.

$$\text{Distance} = \text{Speed} \times \text{Time}$$

If the speed is constant, only multiplication is needed, but if the speed changes, then we need a more powerful method of finding the distance. One such method is to approximate the distance traveled by breaking up the time into many short intervals of time, then multiplying the time elapsed in each interval by one of the speeds in that interval, and then taking the sum (a Riemann sum) of the approximate distance traveled in each interval. The basic idea is that if only a short time elapses, then the speed will stay more or less the same. However, a Riemann sum only gives an approximation of the distance traveled. We must take the limit of all such Riemann sums to find the exact distance traveled.



Integration can be thought of as measuring the area under a curve, defined by $f(x)$, between two points (here a and b).

If $f(x)$ in the diagram on the left represents speed as it varies over time, the distance traveled (between the times represented by a and b) is the area of the shaded region S .

To approximate that area, an intuitive method would be to divide up the distance between a and b into a number of equal segments, the length of each segment represented by the symbol Δx . For each small segment, we can choose one value of the function $f(x)$. Call that value h . Then the area of the rectangle with base Δx and height h gives the distance (time Δx multiplied by speed h) traveled in that segment. Associated with each segment is the average value of the function above it, $f(x) = h$. The sum of all such rectangles gives an approximation of the area between the axis and the curve, which is an approximation of the total distance traveled. A smaller value for Δx will give more

rectangles and in most cases a better approximation, but for an exact answer we need to take a limit as Δx approaches zero.

The symbol of integration is \int , an elongated S (the S stands for “sum”). The definite integral is written as:

$$\int_a^b f(x)dx$$

and is read *the integral from a to b of f of x with respect to x* . The Leibniz notation dx is intended to suggest dividing the area under the curve into an infinite number of rectangles, so that their width Δx becomes the infinitesimally small dx . In a formulation of the calculus based on limits, the notation $\int \dots dx$ is to be understood as an operator that takes a function as an input and gives a number, the area, as an output; dx is not a number, and is not being multiplied by $f(x)$.

The indefinite integral, or antiderivative, is written:

$$\int f(x)dx.$$

Functions differing by only a constant have the same derivative, and therefore the antiderivative of a given function is actually a family of functions differing only by a constant. Since the derivative of the function $y = x^2 + C$, where C is any constant, is $y' = 2x$, the antiderivative of the latter is given by:

$$\int 2x dx = x^2 + C.$$

An undetermined constant like C in the antiderivative is known as a constant of integration.

Fundamental theorem

The fundamental theorem of calculus states that differentiation and integration are inverse operations. More precisely, it relates the values of antiderivatives to definite integrals. Because it is usually easier to compute an antiderivative than to apply the definition of a definite integral, the Fundamental Theorem of Calculus provides a practical way of computing definite integrals. It can also be interpreted as a precise statement of the fact that differentiation is the inverse of integration.

The Fundamental Theorem of Calculus states: If a function f is continuous on the interval $[a, b]$ and if F is a function whose derivative is f on the interval (a, b) , then

$$\int_a^b f(x) dx = F(b) - F(a).$$

Furthermore, for every x in the interval (a, b) ,

$$\frac{d}{dx} \int_a^x f(t) dt = f(x).$$

This realization, made by both Newton and Leibniz, who based their results on earlier work by Isaac Barrow, was key to the massive proliferation of analytic results after their work became known. The fundamental theorem provides an algebraic method of

computing many definite integrals—without performing limit processes—by finding formulas for antiderivatives. It is also a prototype solution of a differential equation. Differential equations relate an unknown function to its derivatives, and are ubiquitous in the sciences.

The Integral

Integration is an important concept in mathematics, specifically in the field of calculus and, more broadly, mathematical analysis. Given a function f of a real variable x and an interval $[a, b]$ of the real line, the integral

$$\int_a^b f(x) dx$$

is defined informally to be the net signed area of the region in the xy -plane bounded by the graph of f , the x -axis, and the vertical lines $x = a$ and $x = b$.

The term *integral* may also refer to the notion of antiderivative, a function F whose derivative is the given function f . In this case it is called an *indefinite integral*, while the integrals discussed in this article are termed *definite integrals*. Some authors maintain a distinction between antiderivatives and indefinite integrals.

The principles of integration were formulated independently by Isaac Newton and Gottfried Leibniz in the late seventeenth century. Through the Fundamental Theorem of Calculus, which they independently developed, integration is connected with differentiation: *if f is a continuous real-valued function defined on a closed interval $[a, b]$, then, once an antiderivative F of f is known, the definite integral of f over that interval is given by*

$$\int_a^b f(x) dx = F(b) - F(a).$$

Integrals and derivatives became the basic tools of calculus, with numerous applications in science and engineering. A rigorous mathematical definition of the integral was given by Bernhard Riemann. It is based on a limiting procedure which approximates the area of a curvilinear region by breaking the region into thin vertical slabs. Beginning in the nineteenth century, more sophisticated notions of integral began to appear, where the type of the function as well as the domain over which the integration is performed has been generalized. A line integral is defined for functions of two or three variables, and the interval of integration $[a, b]$ is replaced by a certain curve connecting two points on the plane or in the space. In a surface integral, the curve is replaced by a piece of a surface in the three-dimensional space. Integrals of differential forms play a fundamental role in modern differential geometry. These generalizations of integral first arose from the needs of physics, and they play an important role in the formulation of many physical laws, notably those of electrodynamics. Modern concepts of integration are based on the abstract mathematical theory known as Lebesgue integration, developed by Henri Lebesgue.

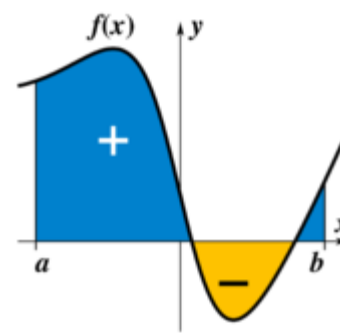


Figure 1: A definite integral of a function can be represented as the signed area of the region bounded by its graph.

History

Pre-calculus integration

Integration may be traced as far back as ancient Egypt, *circa* 1800 BCE. The Moscow Mathematical Papyrus demonstrates a knowledge of the formula for the volume of a pyramidal frustum. The first documented systematic technique, however, capable of determining integrals is the method of exhaustion of Eudoxus (*circa* 370 BC), which sought to find areas and volumes by breaking them up into an infinite number of shapes for which the area or volume was known.

This method was further developed and employed by Archimedes and used to calculate areas for parabolas and an approximation to the area of a circle. His results were usually expressed, not in absolute terms, but in terms of comparisons of volumes. For instance, suppose we have a sphere of radius r which is surrounded exactly by a circular cylinder of radius r and height $2r$. Then Archimedes showed that the volume of the sphere is two thirds that of the cylinder. This is much easier for us with the power of symbolic algebra, for if V is the volume of the sphere and V_1 is the volume of the cylinder, then

$$V = \frac{4}{3} \pi r^3 \text{ and } V_1 = 2\pi r^3$$

and the result follows.

Each case of his area and volume calculations was worked out on its own merits, and no algorithm for handling general problems emerged. It was also important for Archimedes to maintain Greek standards of rigor in proof, and no clear idea of limiting processes or of infinity was known at the time. This made the method rather laborious by modern standards. Some of Archimedes's ideas were known in the Renaissance, as his work *On the Sphere and Cylinder* was available in Latin translations. Interestingly enough, another important contribution of Archimedes, called simply *The Method*, which contained further volume calculations, was not known until 1906. A palimpsest (which is a parchment that has been partly erased and then re-used), on which a copy of *The Method* had been written by a 10th century scribe, was discovered in Constantinople by the historian Joseph Heiberg. It is difficult to read directly, but under ultra-violet light with modern image-enhancing technology it may easily be deciphered. It is the most important Archimedes manuscript known to date, and is the sole source of some of his work. The manuscript was sold at auction in New York in 1998 for around 2 million dollars. The collector deposited the Palimpsest at the Walters Art Museum, Baltimore, for exhibition, conservation, imaging and scholarly study in 1999. Work has been ongoing ever since. More information is available at the palimpsest website <http://www.archimedespalimpsest.org/>.

Similar exhaustion methods were independently developed in China around the 3rd Century AD by Liu Hui, who used it to find the area of the circle. This method was later used in the 5th century by Chinese father and son mathematicians Zu Chongzhi and Zu Geng to find the volume of a sphere. That same century, the Indian mathematician Aryabhata used a similar method in order to find the volume of a cube.

The next major step in integral calculus came in the 11th century, when the Iraqi mathematician, Ibn al-Haytham (known as *Alhazen* in Europe), devised what is now known as *Alhazen's problem*, which leads to an equation of the fourth degree, in his

Book of Optics. While solving this problem, he performed an integration in order to find the volume of a paraboloid. Using mathematical induction, he was able to generalize his result for the integrals of polynomials up to the fourth degree. He thus came close to finding a general formula for the integrals of polynomials, but he was not concerned with any polynomials higher than the fourth degree. Some ideas of integral calculus are also found in the *Siddhanta Shiromani*, a 12th century astronomy text by Indian mathematician Bhāskara II.

Integration Just Prior to Newton and Leibniz

No further progress was made until the 16th century when mechanics began to drive mathematicians to examine problems such as centers of gravity. Luca Valerio (1552-1618) published *De quadratura parabolae* in Rome (1606) which continued the Greek methods of exhaustion of attacking these type of area problems.

The famous German astronomer Johann Kepler (1571-1630) knew of Archimedes's work on volume calculation. He was able to put his knowledge to practical use when he was asked to find the best proportions for making wine casks. His response to this practical problem was the book he wrote in 1615 *Nova stereometria doliorum* (*New solid geometry of wine casks*). In this work, Kepler considered both theoretical and applicable volume calculations. He broke away from the methods of Archimedes, abandoning the extreme rigor previously considered appropriate, and introduced new approximation techniques. One of his main contributions was the volume of the solid obtained by rotating a segment of a conic section about an axis in its own plane. His book gives the volumes of almost 100 solids. He is considered the great precursor of the infinitesimal method of the integral calculus. In his work on planetary motion he had to find the area of sectors of an ellipse. His method consisted of thinking of areas as sums of lines, another crude form of integration. Kepler was rather lucky to obtain the correct answer after making two cancelling errors in this work.

Three mathematicians, born within three years of each other, were the next to make major contributions. They were Fermat, Roberval and Cavalieri. Bonaventura Cavalieri (1598-1647) was led to his *method of indivisibles* by Kepler's attempts at integration. He was not rigorous in his approach and it is hard to see clearly how he thought about his method. It appears that Cavalieri thought of an area as being made up of components which were lines and then summed his infinite number of *indivisibles*. He showed, using these methods, that the integral of x^n from 0 to a was $a^{n+1}/(n + 1)$ by showing the result for a number of values of n and inferring the general result.

$$\int_0^a x^n dx = \frac{a^{n+1}}{n+1}.$$

Cavalieri was a follower of Galileo Galilei, who was himself interested in problems involving area and volume. Cavalieri's ideas are contained in his book *Geometria indivisibilibus continuorum* of 1635. In this work he expounded his method of indivisibles, an infinitesimal technique influenced by Archimedes's method of exhaustion. Cavalieri considered a line as composed of an infinite number of points, a surface as composed of an infinite number of lines, and so on. The idea of an indivisible was not entirely new, as it had occurred already in medieval scholastic philosophy.

The main advantage of the method of indivisibles was that it was more systematic compared with the method of exhaustion, although perhaps more subject to criticism from lacking extreme rigor. The method made it easy to find the area of an ellipse or the volume of a sphere. The main thrust of his arguments was to establish inequalities for estimating the sums

$$1^m + 2^m + \cdots + n^m$$

when n and m are positive integers. In effect, by looking at special cases and generalizing, he established inequalities of the form

$$1^m + 2^m + \cdots + n^m > \frac{n^{m+1}}{m+1} > 1^m + 2^m + \cdots + (n-1)^m.$$

The same type of analysis had in fact already been considered by the French mathematicians Fermat and Roberval a little earlier.

Roberval considered problems of the same type but was more rigorous than Cavalieri. Roberval looked at the area between a curve and a line as being made up of an infinite number of infinitely narrow rectangular strips. He applied this to the integral of x^m from 0 to 1 which he showed had approximate value $(0^m + 1^m + 2^m + \cdots + (n-1)^m)/n^{m+1}$.

$$\int_0^1 x^m dx \approx \frac{0^m + 1^m + 2^m + \cdots + (n-1)^m}{n^{m+1}}.$$

Roberval then asserted that this tended to $1/(m+1)$ as n tends to infinity, so calculating the area.

Fermat was also more rigorous in his approach but gave no proofs. He generalized the parabola and hyperbola to higher powered curves

$$\text{Parabola: } y/a = (x/b)^2 \quad \text{to} \quad (y/a)^n = (x/b)^m$$

$$\text{Hyperbola: } y/a = b/x \quad \text{to} \quad (y/a)^n = (b/x)^m.$$

In the course of examining $y/a = (x/b)^p$, Fermat computed the sum of r^p from $r = 1$ to $r = n$.

Fermat also investigated maxima and minima by considering when the tangent to the curve was parallel to the x -axis. He wrote to Descartes giving the method essentially as used today, namely finding maxima and minima by calculating when the derivative of the function was 0. In fact, because of this work, Lagrange stated clearly that he considers Fermat to be the inventor of the calculus.

Descartes produced an important method of determining normals in *La Géométrie* in 1637 based on double intersection. De Beaune extended his methods and applied it to tangents where double intersection translates into double roots. Hudde discovered a simpler method, known as *Hudde's Rule*, which basically involves the derivative. Descartes' method and Hudde's Rule were important in influencing Newton.

Huygens was critical of Cavalieri's proofs saying that what one needs is a proof which at least convinces one that a rigorous proof could be constructed. Huygens was a major influence on Leibniz and so played a considerable part in producing a more satisfactory approach to the calculus.

The next significant contribution to the methods of the calculus that we wish to describe is that made by the English mathematician John Wallis (1616-1703). Wallis was professor of geometry at Oxford and he wrote a number of influential books. The one of interest in calculus is his *Arithmetica infinitorum*, published in 1655. Wallis knew of the existence of Cavalieri's work on indivisibles, but probably had never seen his manuscript of *Geometria indivisibilibus continuorum*. Wallis obtained results, amounting to the calculation of definite integrals, similar to those of Cavalieri. His approach was more arithmetical and less laborious than Cavalieri's. Indeed, Wallis was probably more of an algebraist than a geometer. On the basis of his preliminary findings, he was able to infer the general pattern for evaluating the integral

$$\int_0^1 x^k dx.$$

He used an argument by analogy, in which he did not rigorously justify all his steps. This led to criticism from the continental mathematicians. Wallis was the first mathematician to make use of fractional powers, such as $x^{1/2}$ or $x^{2/3}$. Descartes, only twenty or so years earlier, had popularized the use of integer powers for exponents. Wallis even went so far as to integrate such fractional powers, obtaining

$$\int_0^1 x^{\frac{p}{q}} dx = \frac{1}{\frac{p}{q} + 1}.$$

His argument was done by analogy with the whole number case, an idea that is still used in introductory courses in integration. Despite the lack of mathematical rigor, his results were correct and provided a powerful impetus for generalization by Newton a few years later.

Wallis also showed the relevance of infinite processes to problems of mathematical analysis. A noteworthy example is his infinite product formula

$$\frac{\pi}{4} = \frac{2 \times 4 \times 4 \times 6 \times 6 \times 8 \times \cdots}{3 \times 3 \times 5 \times 5 \times 7 \times 7 \times \cdots}$$

In our modern notation, he obtained this result by considering integrals of the form

$$\int_0^{\pi/2} \sin^m \theta d\theta.$$

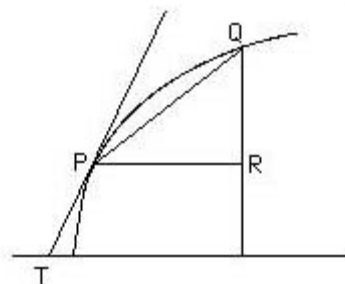
His arguments in obtaining this formula were not entirely rigorous but they proved to be correct, and he had thus extended the range of theorems accessible by limiting processes. The formula for $\pi/4$ is remarkable, as it is transcendental (it is not the root of any polynomial with rational coefficients) but is shown to be a limit of simple fractions. We could say that Wallis prepared the way for the study of infinite series, rather than polynomials. Such series have been a key feature in the calculus and much of all subsequent analysis.

The next major step was provided by Torricelli and Isaac Barrow (1630-77). He was professor of mathematics at Cambridge from 1663 until 1669. His successor in the professorship was Newton. Barrow was primarily a geometer following the traditions of

the ancient Greeks. However, around 1664 he became interested in the problem of finding tangents to curves and he developed an approach involving moving points and lines. In his university lectures at Cambridge, which were subsequently published, he gave his own generalization of tangent and area procedures based on his extensive reading of the works of such notable contemporary mathematicians as Descartes, Wallis, Fermat and especially the Scottish mathematician James Gregory, who is considered to be an important forerunner of Newton. The lectures contained ideas that could have been exploited but they were probably not studied outside Cambridge. Barrow gave a method of tangents to a curve where the tangent is given as the limit of a chord as the points approach each other known as *Barrow's differential triangle*.

Here is Barrow's differential triangle.

Both Torricelli and Barrow considered the problem of motion with variable speed. The derivative of the distance is velocity and the inverse operation takes one from the velocity to the distance. Hence an awareness of the inverse of differentiation began to evolve naturally and the idea that integral and derivative were inverses to each other were familiar to Barrow. In fact, although Barrow never explicitly stated the fundamental theorem of the calculus, he was working towards the result and Newton was to continue with this direction and state the Fundamental Theorem of the Calculus explicitly.



It has been a matter of much conjecture whether Newton was in any sense a student of Barrow's. It was always assumed that he was influenced by Barrow, as he was working at Cambridge at the time of Barrow's lectures on tangent and area problems. Furthermore, Newton's first great advances in the foundations of the calculus date from 1664-65, which is the time when Barrow first studied the problems that underlie the calculus. It is clear that Barrow's notion of generating curves by the motion of points was important to Newton's foundation of the differential calculus, but on the whole Newton denied any direct influence from Barrow.

Newton and Leibniz

We turn now to study the work of the person generally held to be the first to develop the calculus on a systematic basis and see the connection between the differential and integral processes. Isaac Newton (1642-1727) was born on Christmas Day 1642 in Lincolnshire, England. His father had already died in October 1642 and the lack of a father is thought to have had an effect on Newton's personality. Newton lived at Woolsthorpe, near Grantham, in a house which can still be visited. He attended a grammar school in Grantham, where he learned Latin, but little mathematics. His mother intended him to become a farmer, but he showed no aptitude for farm work. Instead, he was sent to Cambridge University in 1661, where he graduated in 1665.

As an undergraduate, he studied Descartes's *La Geometrie* in a Latin translation, as well as Wallis's *Arithmetica infinitorum* and Aristotle's philosophy. He also read Euclid's *Elements* but was unimpressed by it initially. In 1665, the university was closed because

of the plague, and Newton spent most of the next 18 months in Lincolnshire. It was during this time that Newton made four fundamental discoveries:

- the general binomial theorem;
- the connection between the methods used in tangent and area problems (the fundamental theorem of the calculus);
- the law of universal gravitation;
- the theory of the composition of white light.

In 1667, he was elected a Fellow of Trinity College, Cambridge, and in 1669, at the age of 26, he became professor of mathematics at Cambridge University. Isaac Barrow, his predecessor as professor, is said to have resigned in Newton's favor, although he was also seeking advancement in London clerical circles. In the next ten or more years, Newton gave university lectures in optics, arithmetic and algebra, and parts of what became his *Principia*. Much of this work was published subsequently and proved immensely influential, although it is reported that his lectures were largely unattended and unappreciated. In the 1690's, after he had published his fundamental work, he apparently lost interest in mathematics, and eventually left Cambridge to become Master of the Mint in London.

He still, however, found time to revise and publish other parts of his work and engage in scientific correspondence and controversy.

Newton especially studied Descartes's *La Geometrie* and Wallis's *Arithmetica infinitorum*, which strongly influenced his work on analytic geometry and algebra, and on calculus. He also studied the work of Fermat and James Gregory. At the beginning of 1665, Newton discovered the general binomial theorem. He recalled later:

In the winter of the years 1664 and 1665 upon reading Dr Wallis's Arithmetica Infinitorum & trying to interpolate his progressions for squaring the circle I found out another infinite series for squaring the circle and then another for squaring the Hyperbola.

Here, squaring the circle means finding the area of a circle, or, in effect, calculating the integral. Upon reading Wallis's work on finding areas, Newton was led to understand that the integrand $\sqrt{1-x^2}$ could be expanded as an infinite series and the general indefinite integral could also be expressed by term-by-term integration as

$$x - \frac{\frac{1}{2}x^3}{3} - \frac{\frac{1}{8}x^5}{5} - \frac{\frac{1}{16}x^7}{7} - \dots$$

James Gregory had applied similar ideas to evaluating

$$\int_0^1 \sqrt{1-x^2} dx, \int \sqrt{1-x^2} dx, \text{ and } \int \frac{1}{1+x^2} dx$$

by expanding $(1+x^2)^{-1}$ as an infinite geometric series, thereby obtaining an inverse tangent series.

Newton also discovered the infinite series representation for the $\sin^{-1} x$ function, and for

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

He used his formula to calculate special values of the log function, giving areas under a hyperbola, to fifty or more decimal places. The surviving manuscripts show that Newton was a formidable calculator, well able to test the accuracy of his theoretical results. Around the same time, Newton developed a differentiation procedure based on the concept of an infinitesimally small increase o , which ultimately vanishes, of a variable x . Eventually, he settled on the notion of a fluxion of the variable, meaning a finite instantaneous speed defined with respect to an independent time variable. In our terms, the fluxion of x with respect to the time variable t is $\frac{dx}{dt}$, which is the speed in

physical terms. Later, he introduced the notation \dot{x} for $\frac{dx}{dt}$, \ddot{x} for $\frac{d^2x}{dt^2}$, and so on, which has been maintained to some extent in dynamics. He also invented the idea of partial derivatives of a function of two independent variables in 1665, although again, his notation was very different from that which evolved later.

Between 1664 and 1666, Newton was immensely creative mathematically, but for the next two years his interest moved elsewhere. In 1669, he saw a copy of Mercator's book *Logarithmotechnia*, published in late 1668, in which the infinite series for $\log(1+x)$ was given. Newton suddenly realized that other people were discovering the method of infinite series, which he had greatly developed in 1664-65. So he wrote a tract, entitled *De analysi per aequationes infinitas*. This tract was circulated to several important mathematicians, to establish Newton's priority of discovery. The tract was handwritten and was never printed, although a version subsequently appeared many years later, incorporated into a larger work. It was perhaps typical of Newton that he never really gave a proper presentation of his work on the differential calculus, and this was partly to blame for the famous dispute with Leibniz.

In his *De analysi* manuscript, Newton was able to show that the area under the curve $y = x^{m/n}$ is given by the function

$$\frac{x^{1+m/n}}{1+m/n}$$

by introducing an area function and considering small increments in this function. He was thus able to demonstrate that the area problem was the inverse of the tangent problem for curves. While special cases of this property were evident in the work of such mathematicians as James Gregory, Newton was the first to exploit it systematically, especially through his use of infinite series.

Newton changed his approach to the limiting processes over the years. However, as the limit concept is subtle, and was not properly clarified until the mid-19th century, Newton was always vulnerable to the accusation that his arguments relied on the manipulation of quantities that are ultimately set equal to 0, and that he was effectively dividing by 0.

Newton never published a self-contained account of his theory of the calculus. In his masterpiece *Philosophiae naturalis principia mathematica* (*Mathematical principles of natural philosophy*), published in 1687, he made use of some of his ideas of calculating derivatives, although most of the emphasis is given to geometric rather than algebraic

methods. (It was once thought that Newton had an aversion to the use of algebra, especially in his published work, but existing manuscripts show that he freely used algebra in his calculations.) The *Principia* (as it is generally named) is mainly concerned with physics and astronomy, and its intent is to show how the laws of motion and the inverse square law of gravitation enable us to describe the workings of the solar system very accurately. It is of interest to note that, in the first two editions of the *Principia*, Newton acknowledged that Leibniz had independently discovered a version of the calculus, but following the acrimony of their dispute, all reference to Leibniz had been removed from the third edition of 1726.

This major advance in integration came in the 17th century with the independent discovery of the fundamental theorem of calculus by Newton and Leibniz. The theorem demonstrates a connection between integration and differentiation. This connection, combined with the comparative ease of differentiation, can be exploited to calculate integrals. In particular, the fundamental theorem of calculus allows one to solve a much broader class of problems. Equal in importance is the comprehensive mathematical framework that both Newton and Leibniz developed. Given the name infinitesimal calculus, it allowed for precise analysis of functions within continuous domains. This framework eventually became modern calculus, whose notation for integrals is drawn directly from the work of Leibniz.

Newton wrote a tract on *fluxions* in October 1666. This was a work which was not published at the time but seen by many mathematicians and had a major influence on the direction the calculus was to take. Newton thought of a particle tracing out a curve with two moving lines which were the coordinates. The horizontal velocity x' and the vertical velocity y' were the fluxions of x and y associated with the flux of time. The fluents or *flowing quantities* were x and y themselves. With this fluxion notation y'/x' was the tangent to $f(x,y) = 0$.

In his 1666 tract Newton discusses the converse problem, given the relationship between x and y'/x' , find y . Hence the slope of the tangent was given for each x and when $y'/x' = f(x)$ then Newton solves the problem by antidifferentiation. He also calculated areas by antidifferentiation and this work contains the first clear statement of the *Fundamental Theorem of the Calculus*.

Newton had problems publishing his mathematical work. Barrow was in some way to blame for this since the publisher of Barrow's work had gone bankrupt and publishers were, after this, wary of publishing mathematical works. Newton's work on *Analysis with infinite series* was written in 1669 and circulated in manuscript form. It was not published until 1711. Similarly his *Method of fluxions and infinite series* was written in 1671 and published in English translation in 1736. The Latin original was not published until much later.

In these two works Newton calculated the series expansion for $\sin x$ and $\cos x$ and the expansion for what was actually the exponential function, although this function was not established until Euler introduced the present notation e^x .

Newton's next mathematical work was *Tractatus de Quadratura Curvarum* which he wrote in 1693 but it was not published until 1704 when he published it as an Appendix

to his *Optiks*. This work contains another approach which involves taking limits. Newton says

In the time in which x by flowing becomes $x+o$, the quantity x^n becomes $(x+o)^n$ i.e. by the method of infinite series,

$$x^n + nox^{n-1} + (nn-n)/2 oox^{n-2} + \dots$$

At the end he lets the increment o vanish by 'taking limits'.

Leibniz learned much on a European tour on which he met Huygens in Paris in 1672. He also met Hooke and Boyle in London in 1673 where he bought several mathematics books, including Barrow's works. Leibniz was to have a lengthy correspondence with Barrow. On returning to Paris Leibniz did some very fine work on the calculus, thinking of the foundations very differently from Newton.

Newton considered variables changing with time. Leibniz thought of variables x , y as ranging over sequences of infinitely close values. He introduced dx and dy as differences between successive values of these sequences. Leibniz knew that dy/dx gives the tangent but he did not use it as a defining property.

For Newton integration consisted of finding *fluents* for a given fluxion so the fact that integration and differentiation were inverses was implied. Leibniz used integration as a sum, in a way similar to Cavalieri. He was also happy to use 'infinitesimals' dx and dy where Newton used x' and y' which were finite velocities. Of course neither Leibniz nor Newton thought in terms of functions, however, but both always thought in terms of graphs. For Newton the calculus was **geometrical** while Leibniz took it towards **analysis**.

Leibniz was very conscious that finding a good notation was of fundamental importance and thought a lot about it. Newton, on the other hand, wrote more for himself and, as a consequence, tended to use whatever notation he thought of on the day. Leibniz's notation of d and \int highlighted the operator aspect which proved important in later developments. By 1675 Leibniz had settled on the notation

$$\int y dy = y^2/2$$

written exactly as it would be today. His results on the integral calculus were published in 1684 and 1686 under the name '*calculus summatorius*', the name integral calculus was suggested by Jacob Bernoulli in 1690.

Gottfried Wilhelm Leibniz (1646-1716) was born in Leipzig, in Germany. At university, he studied a wide range of topics, including law, theology, philosophy and mathematics. Like Descartes, he was determined to find a universal procedure of reasoning, applicable to virtually all academic disciplines. (He was interested in inventing an algebra of logic, something that George Boole later introduced in his book *The Laws of Thought*, published in 1854.) This abstract approach to reasoning probably ensured that his version of the calculus was systematic and algorithmic, unlike Newton's. Much of Leibniz's career was spent in the diplomatic service of the rulers of Hanover, which enabled him to visit many of the intellectual centers of Europe, including Paris in the early 1670's. He visited London in 1673 and 1676 and probably saw the manuscript of Newton's *De analysi* during his stays.

Around 1673, Leibniz realized that the determination of the tangent to a curve depended on the ratio of the difference of the y -values (the ordinates) to the difference of the x -values (abscissae), as these differences become infinitesimally small. He also realized that the area under a curve is found by taking the sum of the areas of infinitely thin rectangles. Like Newton, he was led to see the role of infinite series (possibly influenced by Newton's *De analysi*). By 1676 he had essentially obtained all of Newton's slightly earlier conclusions. Furthermore he saw that his procedures could deal not only with simple powers of x but also with so-called transcendental functions, such as $\log x$ and $\sin x$.

We owe to Leibniz much of the familiar notation of the calculus, and it is generally agreed that Leibniz's exposition of the calculus and its fundamental processes was much clearer and more easily used than that of Newton. He eventually decided on the use of dx and dy for smallest possible x and y increases (they are sometimes called differentials). Likewise, he introduced $\int y$ and $\int y dx$ for finding area, the integral sign arising as an enlarged s , signifying sum. The *calculus differentialis* became the method for finding tangents and the *calculus summatorius* or *calculus integralis* the method for finding areas.

Leibniz was the first person to publish a complete account of the differential calculus. His paper was entitled *Nova methodus pro maximis et minimis, itemque tangentibus*. It appeared in an important journal, *Acta Eruditorum*, published in Leipzig in 1684. This journal was a major source of news in science, where reviews of such books as Newton's *Principia* kept its readers up to date with the latest ideas. In his paper, Leibniz used the d notation for derivatives (or differentials). He stated quite clearly the product rule in the form

$$d\overline{xy} = xdy + ydx$$

where the bar denotes product, and the quotient rule as

$$d\frac{v}{y} = \frac{vdy - ydv}{y^2}$$

He also gave the rule for differentiating powers as

$$dx^n = nx^{n-1} dx.$$

In 1686, in the same journal, Leibniz gave an account of the integral calculus, emphasizing the inverse nature of the procedures of integration and differentiation. Leibniz's presentation of the differential calculus proved to be influential, as he soon found followers such as the Bernoulli brothers, who were able to extend the range of its applications significantly.

It may be of interest to examine what communication existed between Newton and Leibniz in the years when Leibniz was perfecting his differential calculus. In 1676, in answer to a request from Leibniz, Newton wrote to him to give brief details of his theory of the calculus. On August 27 of the same year, Leibniz wrote back to him to ask for fuller details on this subject. Newton replied on October 24 1676 in a long letter, describing how he had been led to some of his discoveries. He had interpolated the

results of Wallis, and, working by analogy, saw how to expand functions such as $(1 - x^2)^{\frac{1}{2}}$ and $(1 - x^2)^{\frac{3}{2}}$ into infinite series. He had then proceeded to deduce (or guess) the general binomial theorem. In this same letter, Newton mentioned his method of fluxions, but he gave no details. He also listed some of the functions which he could integrate, enabling him to find corresponding areas. These included $x^m(b + cx^n)^p$ and $x^{m-1}(a + b + cx^{2n})^{\pm\frac{1}{2}}$.

Newton concluded by expressing regret about controversy arising from his earlier research publications, with the implication that he intended to publish little in future. Leibniz replied on June 21, 1677. He described his method of drawing tangents, which proceeded not by fluxions of lines but by differences of numbers. He also introduced his notation of dx and dy for infinitesimal differences or differentials of coordinates.

While Leibniz's differential calculus seems quite familiar to modern day students of mathematics, the same cannot be said of Newton's fluxional calculus, which appears unnecessarily complicated. Let's briefly attempt to describe how Newton's calculus proceeded. He assumed that we may conceive of all geometric magnitudes as generated by continuous motion – for example, a line is generated by the motion of a point, *etc.* The quantity thus generated is called the fluent or flowing quantity. The velocity of the moving magnitude is the fluxion of the fluent. There then arise two problems. The first is to find the fluxion of a given quantity, or more generally the relation of the fluents being given, to find the relation of their fluxions. This is akin to implicit differentiation. The second method is the inverse method of fluxions: from the fluxion, or some relation involving it, to find the fluent. This amounts to what we call integration, possibly of a differential equation. Newton referred to these procedures as the method of quadrature and the inverse method of tangents. The infinitely small part by which a fluent such as x increased in a small interval of time o was called the moment of the fluent and its value was shown to be ox . Given x Newton denoted the fluent whose fluxion was x by x' or $[x]$. This is the same as the integral of x (but not with respect to x). Subsequently, Newton's methods were taught at Cambridge for more than 100 years, and the word fluxion was retained. The explanations given for the procedures were usually vague and unconvincing. It was not until almost 1820 that Leibniz's notation of dx and the integral sign started to appear in British mathematical textbooks and papers.

Between 1689 and 1693, Newton had fallen under the influence of a little-known Swiss mathematician named Nicholas Fatio de Duillier (1664-1753), who had moved to England in 1687. Later in 1693, Newton suffered what appears to have been a nervous breakdown. After his recovery, Newton decided to abandon his career in Cambridge, devoted to research and scientific discovery, and sought more worldly fame in London as Warden of the Mint in 1696. Nonetheless, Newton was alarmed when informed by John Wallis in 1695 that many continental mathematicians considered the calculus to be the invention of Leibniz alone, perhaps on the basis of his expositions of the subject in *Ada Eruditorum*. In support of Newton, Duillier published a paper through the Royal Society of London in 1699 in which he implied that Leibniz had plagiarized the ideas of the calculus from Newton. In 1704 Leibniz replied in the *Ada Eruditorum* that he had priority of publication, and he protested to the Royal Society about the unfairness of the accusation of plagiarism. The Royal Society eventually responded by establishing a committee to investigate the dispute. In 1712 the committee published their conclusions

in a report entitled *Commercium epistolicum*. The report affirmed that Newton had invented the calculus, a point not seriously disputed even by Leibniz. The report also noted that Leibniz had had access in the 1670's to manuscripts and letters describing Newton's preliminary version of the calculus (for example, *De analysi*), so that suspicions of plagiarism were not totally dismissed. It has become apparent that the *Commercium epistolicum* was essentially Newton's own work — he dictated the conclusions that the committee reported.

The priority dispute was dominated by nationalistic concerns, British mathematicians being especially keen to defend Newton's honor and proclaim his genius as a reflection of British superiority. As a consequence, the advantages of Leibniz's notation, subsequently developed by the Bernoulli's and Euler, were ignored in Britain, and by paying too much deference to Newton, mathematics stagnated there until the early 19th century. The dispute was led in Britain, albeit secretly, by Newton himself, and he seems to have wanted to remove all record of Leibniz's achievements.

After Newton and Leibniz the development of the calculus was continued by Jacob Bernoulli and Johann Bernoulli. When Berkeley published his *Analyst* in 1734 attacking the lack of rigor in the calculus and disputing the logic on which it was based much effort was made to tighten the reasoning. Maclaurin attempted to put the calculus on a rigorous geometrical basis but the really satisfactory basis for the calculus had to wait for the work of Cauchy in the 19th century.

Formalizing integrals

Calculus acquired a firmer footing with the development of limits and was given a suitable foundation by Cauchy in the first half of the 19th century. Integration was first rigorously formalized, using limits, by Riemann. All bounded piecewise continuous functions are Riemann integrable on a bounded interval. More general functions were considered, however, to which Riemann's definition does not apply, and Lebesgue formulated a different definition of integral, founded in measure theory (a subfield of real analysis) which agreed with Riemann's integral on the class of Riemann integrable functions. Other definitions of integral, extending Riemann's and Lebesgue's approaches, have been proposed and used to great advantage.

Fourier (1768-1830) studied heat conduction with a series of trigonometric terms to represent functions. Fourier series and integral transforms have applications today in fields as far apart as medicine, linguistics, and music. This led to a firmer desire to have a rigorous definition of function, as his *functions* did not satisfy the current functional definition. The modern notation for the definite integral, with limits above and below the integral sign, was first used by Joseph Fourier in *Mémoires* of the French Academy around 1819–20 and reprinted in his book of 1822.

Gauss (1777-1855) made the first table of integrals, and with many others continued to apply integrals in the mathematical and physical sciences. Cauchy (1789-1857) took integrals to the complex domain. Riemann (1826-1866) and Lebesgue (1875-1941) put definite integration on a firm logical foundation.

Liouville (1809-1882) created a framework for constructive integration by finding out when indefinite integrals of elementary functions are again elementary functions.

Hermite (1822-1901) found an algorithm for integrating rational functions. In the 1940s Ostrowski extended this algorithm to rational expressions involving the logarithm.

In the 20th century before computers, mathematicians developed the theory of integration and applied it to write tables of integrals and integral transforms. Among these mathematicians were Watson, Titchmarsh, Barnes, Mellin, Meijer, Grobner, Hofreiter, Erdelyi, Lewin, Luke, Magnus, Apelblat, Oberhettinger, Gradshteyn, Ryzhik, Exton, Srivastava, Prudnikov, Brychkov, and Marichev.

In 1969 Risch made the major breakthrough in algorithmic indefinite integration when he published his work on the general theory and practice of integrating elementary functions. His algorithm does not automatically apply to all classes of elementary functions because at the heart of it there is a hard differential equation that needs to be solved. Efforts since then have been directed at handling this equation algorithmically for various sets of elementary functions. These efforts have led to an increasingly complete algorithmization of the Risch scheme. In the 1980s some progress was also made in extending his method to certain classes of special functions.

The capability for definite integration gained substantial power in computer algebra systems first released in 1988. Comprehensiveness and accuracy have been given strong consideration in the development of *Maple* and *Mathematica* and have been successfully accomplished in different ways in their integration code. Besides being able to replicate most of the results from well-known collections of integrals (and to find scores of mistakes and typographical errors in them), a good computer algebra system makes it possible to calculate countless new integrals not included in any published handbook

Terminology and notation

If a function has an integral, it is said to be integrable. The function for which the integral is calculated is called the integrand. The region over which a function is being integrated is called the domain of integration. If the integral does not have a domain of integration, it is considered indefinite (one with a domain is considered definite). In general, the integrand may be a function of more than one variable, and the domain of integration may be an area, volume, a higher dimensional region, or even an abstract space that does not have a geometric structure in any usual sense.

The simplest case, the integral of a real-valued function f of one real variable x on the interval $[a, b]$, is denoted by

$$\int_a^b f(x) dx.$$

The \int sign, an elongated “s”, represents integration; a and b are the lower limit and upper limit of integration, defining the domain of integration; f is the integrand, to be evaluated as x varies over the interval $[a, b]$; and dx is the variable of integration.

The variable of integration dx has different interpretations depending on the theory being used. For example, it can be seen as strictly a notation indicating that x is a dummy variable of integration, as a reflection of the weights in the Riemann sum, a measure (in Lebesgue integration and its extensions), an infinitesimal (in non-standard

analysis) or as an independent mathematical quantity as in a *differential form*. More complicated cases may vary the notation slightly.

After the failure of early efforts to rigorously interpret infinitesimals, Riemann formally defined integrals as a limit of weighted sums, so that the dx suggested the limit of a difference (namely, the interval width). Shortcomings of Riemann's dependence on intervals and continuity motivated newer definitions, especially the Lebesgue integral, which is founded on an ability to extend the idea of "measure" in much more flexible ways. Thus the notation

$$\int_A f(x) d\mu$$

refers to a weighted sum in which the function values are partitioned, with μ measuring the weight to be assigned to each value. Here A denotes the region of integration.

Differential geometry, with its *calculus on manifolds*, gives the familiar notation yet another interpretation. Here $f(x)$ and dx become a differential form, $\omega = f(x) dx$, a new differential operator \mathbf{d} , known as the *exterior derivative* appears, and the fundamental theorem becomes the more general Stokes' theorem,

$$\int_A d\omega = \int_{\partial A} \omega,$$

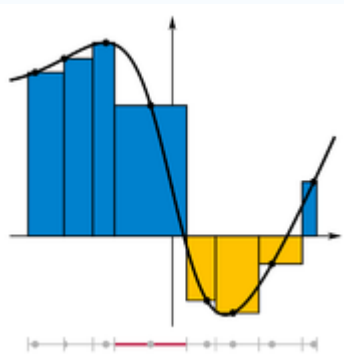
from which Green's theorem, the divergence theorem, and the fundamental theorem of calculus follow.

Although there are differences between these conceptions of integral, there is considerable overlap. Thus the area of the surface of the oval swimming pool can be handled as a geometric ellipse, as a sum of infinitesimals, as a Riemann integral, as a Lebesgue integral, or as a manifold with a differential form. The calculated result will be the same for all.

Formal definitions

There are many ways of formally defining an integral, not all of which are equivalent. The differences exist mostly to deal with differing special cases which may not be integrable under other definitions, but also occasionally for pedagogical reasons. The most commonly used definitions of integral are Riemann integrals and Lebesgue integrals.

Riemann integral



The Riemann integral is defined in terms of Riemann sums of functions with respect to *chosen partitions* of an interval. Let $[a,b]$ be a closed interval of the real line; then a *chosen partition* of $[a,b]$ is a finite sequence

$$a = x_0 \leq t_1 \leq x_1 \leq t_2 \leq x_2 \leq \cdots \leq x_{n-1} \leq t_n \leq x_n = b.$$

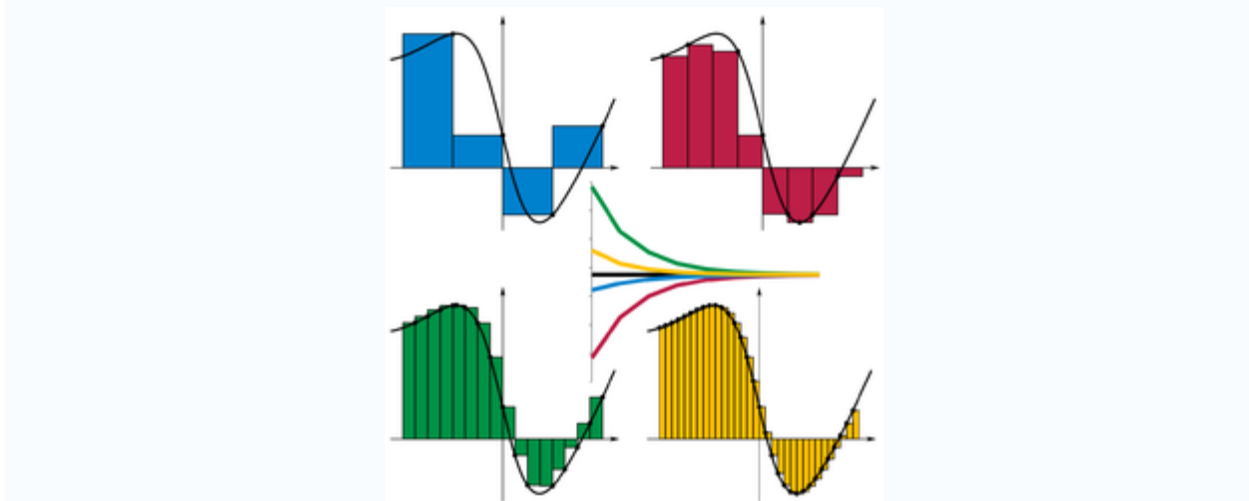


Figure 2: Riemann sums converging as intervals halve, whether sampled at ■ right, ■ minimum, ■ maximum, or ■ left.

This sequence partitions the interval $[a,b]$ into i sub-intervals $[x_{i-1}, x_i]$, each of which is *tagged* with a distinguished point $t_i \in [x_{i-1}, x_i]$. Let $\Delta_i = x_i - x_{i-1}$ be the width of the i th sub-interval. Then the *mesh* of such a chosen partition is the width of the largest sub-interval formed by the partition, $\max_{i=1,\dots,n} \Delta_i$. A *Riemann sum* of a function f with respect to such a chosen partition is defined as

$$\sum_{i=1}^n f(t_i) \Delta_i.$$

Thus each term of the sum is the area of a rectangle with height equal to the function value at the distinguished point of the given sub-interval, and width the same as the sub-interval width. The *Riemann integral* of a function f over the interval $[a,b]$ is equal to S if:

For any $\epsilon > 0$ there exists $\delta > 0$ such that, for any chosen partition $[a,b]$ with mesh less than δ , we have

$$\left| S - \sum_{i=1}^n f(t_i) \Delta_i \right| < \epsilon.$$

When the chosen tags give the maximum (respectively, minimum) value of each interval, the Riemann sum becomes an upper (respectively, lower) Darboux sum, suggesting the close connection between the Riemann integral and the Darboux integral.

Lebesgue integral

The Riemann integral is not defined for a wide range of functions and situations of importance in applications (and of interest in theory). This motivates other definitions, under which a broader assortment of functions is integrable. The Lebesgue integral, in

particular, achieves great flexibility by directing attention to the weights in the weighted sum.

The definition of the Lebesgue integral begins with a measure, μ . In the simplest case the Lebesgue measure $\mu(A)$ of an interval $A = [a, b]$ is its width, $b - a$, so that the Lebesgue integral agrees with the (proper) Riemann integral when both exist. In more complicated cases, the sets being measured can be highly fragmented, with no connectivity and no resemblance to intervals.

One approach first defines the integral of the indicator function of a measurable set A simply by:

$$\int 1_A d\mu = \mu(A)$$

where 1_A is the characteristic function of the set A :

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}.$$

This extends by linearity to a measurable simple function s , which attains only a finite number, n , of distinct non-negative values:

$$\begin{aligned} \int s d\mu &= \int \left(\sum_{i=1}^n a_i 1_{A_i} \right) d\mu \\ &= \sum_{i=1}^n a_i \int 1_{A_i} d\mu \\ &= \sum_{i=1}^n a_i \mu(A_i) \end{aligned}$$

(where the image of A_i under the simple function s is the constant value a_i). Thus if E is a measurable set one defines

$$\int_E s d\mu = \sum_{i=1}^n a_i \mu(A_i \cap E).$$

Then for any non-negative measurable function f one defines

$$\int_E f d\mu = \sup \left\{ \int_E s d\mu \mid 0 \leq s \leq f \text{ and } s \text{ is a simple function} \right\};$$

that is, the integral of f is set to be the supremum of all the integrals of simple functions that are less than or equal to f . A general measurable function f , is split into its positive and negative values by defining

$$f^+(x) = \begin{cases} f(x) & \text{if } f(x) > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f^-(x) = \begin{cases} -f(x) & \text{if } f(x) < 0 \\ 0 & \text{otherwise} \end{cases}$$

Finally, f is Lebesgue integrable if

$$\int_E |f| d\mu < \infty,$$

and then the integral is defined by

$$\int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu.$$

Properties of integration

Linearity

$$\bullet \int_a^b (\alpha f + \beta g)(x) dx = \alpha \int_a^b f(x) dx + \beta \int_a^b g(x) dx$$

Inequalities for integrals

A number of general inequalities hold for Riemann-integrable functions defined on a closed and bounded interval $[a, b]$ and can be generalized to the Lebesgue integral.

- *Upper and lower bounds.* An integrable function f on $[a, b]$, is necessarily bounded on that interval. Thus there are real numbers m and M so that $m \leq f(x) \leq M$ for all x in $[a, b]$. Since the lower and upper sums of f over $[a, b]$ are therefore bounded by, respectively, $m(b - a)$ and $M(b - a)$, it follows that

$$m(b - a) \leq \int_a^b f(x) dx \leq M(b - a)$$

- *Inequalities between functions.* If $f(x) \leq g(x)$ for each x in $[a, b]$ then each of the upper and lower sums of f is bounded above by the upper and lower sums, respectively, of g . Thus

$$\int_a^b f(x) dx \leq \int_a^b g(x) dx$$

- *Subintervals.* If $[c, d]$ is a subinterval of $[a, b]$ and $f(x)$ is non-negative for all x , then

$$\int_c^d f(x) dx \leq \int_a^b f(x) dx$$

- *Products and absolute values of functions.* If f and g are two functions then we may consider their pointwise products and powers, and absolute values:

$$(fg)(x) = f(x)g(x), f^2(x) = (f(x))^2, |f|(x) = |f(x)|$$

If f is Riemann-integrable on $[a, b]$ then the same is true for $|f|$, and

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx.$$

Moreover, if f and g are both Riemann-integrable then f^2 , g^2 , and fg are also Riemann-integrable, and

$$\left(\int_a^b (fg)(x) dx \right)^2 \leq \left(\int_a^b f^2(x) dx \right) \left(\int_a^b g^2(x) dx \right)$$

This inequality, known as the Cauchy–Schwarz inequality.

- *Hölder's inequality.* Suppose that p and q are two real numbers, $1 \leq p, q \leq \infty$ with $1/p + 1/q = 1$, and f and g are two Riemann-integrable functions. Then the functions $|f|^p$ and $|g|^q$ are also integrable and the following Hölder's inequality holds:

$$\left| \int_a^b (fg)(x) dx \right| \leq \left(\int_a^b |f(x)|^p dx \right)^{1/p} \left(\int_a^b |g(x)|^q dx \right)^{1/q}$$

- For $p = q = 2$, Hölder's inequality becomes the Cauchy–Schwarz inequality.
- *Minkowski inequality.* Suppose that $p \geq 1$ is a real number and f and g are Riemann-integrable functions. Then $|f|^p$, $|g|^p$ and $|f + g|^p$ are also Riemann integrable and the following Minkowski inequality holds:

$$\left(\int_a^b |f(x) + g(x)|^p dx\right)^{1/p} \leq \left(\int_a^b |f(x)|^p dx\right)^{1/p} + \left(\int_a^b |g(x)|^p dx\right)^{1/p}$$

Conventions

In this section f is a real-valued Riemann-integrable function. The integral $\int_a^b f(x)dx$ over an interval $[a, b]$ is defined if $a < b$. This means that the upper and lower sums of the function f are evaluated on a partition $a = x_0 \leq x_1 \leq \dots \leq x_n = b$ whose values x_i are increasing. Geometrically, this signifies that integration takes place “left to right”, evaluating f within intervals $[x_i, x_{i+1}]$ where an interval with a higher index lies to the right of one with a lower index. The values a and b , the end-points of the interval, are called the limits of integration of f .

- *Reversing limits of integration.* If $a > b$ then define

$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

This, with $a = b$, implies:

- *Integrals over intervals of length zero.* If a is a real number then

$$\int_a^a f(x)dx = 0.$$

The first convention is necessary in consideration of taking integrals over subintervals of $[a, b]$; the second says that an integral taken over a degenerate interval, or a point, should be zero. One reason for the first convention is that the integrability of f on an interval $[a, b]$ implies that f is integrable on any subinterval $[c, d]$, but in particular integrals have the property that:

- *Additivity of integration on intervals.* If c is any element of $[a, b]$, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Fundamental theorem of calculus

The *fundamental theorem of calculus* is the statement that differentiation and integration are inverse operations: if a continuous function is first integrated and then differentiated, the original function is retrieved. An important consequence, sometimes called the *second fundamental theorem of calculus*, allows one to compute integrals by using an antiderivative of the function to be integrated.

Statements of theorems

First Fundamental Theorem of Calculus. Let f be a real-valued integrable function defined on a closed interval $[a, b]$. If F is defined for x in $[a, b]$ by

$$F(x) = \int_a^x f(t)dt$$

then F is continuous on $[a, b]$. If f is continuous at x in $[a, b]$, then F is differentiable at x , and $F'(x) = f(x)$.

Second Fundamental Theorem of Calculus. Let f be a real-valued integrable function defined on a closed interval $[a, b]$. If F is a function such that $F'(x) = f(x)$ for all x in $[a, b]$ (that is, F is an antiderivative of f), then

$$\int_a^b f(t)dt = F(b) - F(a).$$

Corollary. If f is a continuous function on $[a, b]$, then f is integrable on $[a, b]$, and F , defined by

$$F(x) = \int_a^x f(t)dt$$

is an anti-derivative of f on $[a, b]$. Moreover,

$$\int_a^b f(t)dt = F(b) - F(a).$$

Methods

Computing integrals

The most basic technique for computing definite integrals of one real variable is based on the Fundamental Theorem of Calculus. It proceeds like this:

1. Let $f(x)$ be the function of x to be integrated over a given interval $[a, b]$.
2. Find an antiderivative of f , that is, a function F such that $F' = f$ on the interval.
3. Then, by the fundamental theorem of calculus, provided the integrand and integral have no singularities on the path of integration,

$$\int_a^b f(x)dx = F(b) - F(a).$$

Note that the integral is not actually the antiderivative, but the fundamental theorem allows us to use antiderivatives to evaluate definite integrals.

The difficult step is often finding an antiderivative of f . It is rarely possible to glance at a function and write down its antiderivative. More often, it is necessary to use one of the many techniques that have been developed to evaluate integrals. Most of these techniques rewrite one integral as a different one which is hopefully more tractable. Techniques include:

- Integration by substitution (reverse to the Chain Rule)
- Integration by parts (reverse to the Product Rule)
- Integration by trigonometric substitution
- Integration by partial fractions
- Integration by reduction formulae (usually derived from Integration by Parts)
- Integration using parametric derivatives

Suppose we want to find the integral $\int_0^{\infty} x^2 e^{-3x} dx$. We may solve this by starting with the integral:

$$\int_0^{\infty} e^{-tx} dx = \left[\frac{e^{-tx}}{-t} \right]_0^{\infty} = \left(\lim_{x \rightarrow \infty} \frac{e^{-tx}}{-t} \right) - \left(\frac{e^0}{-t} \right) = 0 - \left(\frac{1}{-t} \right) = \frac{1}{t}.$$

Find the second derivative with respect to t :

$$\frac{d^2}{dt^2} \int_0^{\infty} e^{-tx} dx = \frac{d^2}{dt^2} \frac{1}{t}$$

$$\int_0^{\infty} \frac{d^2}{dt^2} e^{-tx} dx = \frac{d^2}{dt^2} \frac{1}{t}$$

$$\int_0^{\infty} \frac{d}{dt} (-xe^{-tx}) dx = \frac{d}{dt} \left(-\frac{1}{t^2} \right)$$

$$\int_0^{\infty} x^2 e^{-tx} dx = \frac{2}{t^3}$$

Note that this takes the same form as the original proposed question with $t = 3$. Substituting that into our new solution equation:

$$\int_0^{\infty} x^2 e^{-3x} dx = \frac{2}{3^3} = \frac{2}{27}.$$

- Integration using Euler's formula

Consider the integral $\int \sin^2 x dx$. Normally this requires a half-angle substitution:

$\sin^2 x = \frac{1 - \cos 2x}{2}$. Here, we can use Euler's identity instead:

$$\begin{aligned} \int \sin^2 x dx &= \int \left(\frac{e^{ix} - e^{-ix}}{2i} \right)^2 dx \\ &= -\frac{1}{4} \int (e^{2ix} - 2 + e^{-2ix}) dx \\ &= -\frac{1}{4} \left(\frac{e^{2ix}}{2i} - 2x - \frac{e^{-2ix}}{2i} \right) + C \\ &= \frac{1}{4} (2x - \sin 2x) + C \end{aligned}$$

Now, consider the integral $\int \sin^2 x \cos 4x dx$. This is a bear using trigonometric identities, but Euler's method does the following:

$$\begin{aligned}
\int \sin^2 x \cos 4x \, dx &= \int \left(\frac{e^{ix} - e^{-ix}}{2i} \right)^2 \left(\frac{e^{4ix} + e^{-4ix}}{2} \right) dx \\
&= -\frac{1}{8} \int (e^{2ix} - 2 + e^{-2ix})(e^{4ix} + e^{-4ix}) dx \\
&= -\frac{1}{8} \int (e^{6ix} - 2e^{4ix} + e^{2ix} + e^{-2ix} - 2e^{-4ix} + e^{-6ix}) dx \\
&= -\frac{1}{8} \left(\frac{1}{6i}(e^{6ix} - e^{-6ix}) - \frac{2}{4i}(e^{4ix} - e^{-4ix}) + \frac{1}{2i}(e^{2ix} - e^{-2ix}) \right) + C \\
&= -\frac{1}{24} \left(\frac{e^{6ix} - e^{-6ix}}{2i} \right) + \frac{1}{8} \left(\frac{e^{4ix} - e^{-4ix}}{2i} \right) - \frac{1}{8} \left(\frac{e^{2ix} - e^{-2ix}}{2i} \right) + C \\
&= -\frac{1}{24} \sin 6x + \frac{1}{8} \sin 4x - \frac{1}{8} \sin 2x + C
\end{aligned}$$

- Differentiation under the integral sign

This is a technique made more famous by the physicist Richard Feynman. Because of this, many physicists like to make use of this technique. It works on certain types of integrals. Consider:

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{(a \cos^2 x + b \sin^2 x)^2} dx,$$

where both $a, b > 0$. Normally, we would not try this integration, but by differentiating under the integral sign we can. First we will find:

$$J = \int_0^{\frac{\pi}{2}} \frac{1}{a \cos^2 x + b \sin^2 x} dx.$$

Divide both the numerator and the denominator by $\cos^2 x$.

$$\begin{aligned}
J &= \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{a + b \tan^2 x} dx \\
&= \frac{1}{b} \int_0^{\frac{\pi}{2}} \frac{1}{\left(\sqrt{a/b}\right)^2 + \tan^2 x} d(\tan x) \\
&= \frac{1}{\sqrt{ab}} \arctan \left(\sqrt{\frac{b}{a}} \tan x \right) \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2\sqrt{ab}}
\end{aligned}$$

Since the limits of integration are independent of a , we can find the derivative of J with respect to a .

$$J = \int_0^{\frac{\pi}{2}} \frac{1}{a \cos^2 x + b \sin^2 x} dx$$

$$\frac{\partial J}{\partial a} = - \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{(a \cos^2 x + b \sin^2 x)^2} dx$$

Now, since $J = \frac{\pi}{2\sqrt{ab}}$, then

$$\frac{\partial J}{\partial a} = -\frac{\pi}{4\sqrt{a^3b}}.$$

Equating these two relations then yields

$$\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{(a \cos^2 x + b \sin^2 x)^2} dx = \frac{\pi}{4\sqrt{a^3b}}.$$

Now, doing the same with b as we did with a , we get

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{(a \cos^2 x + b \sin^2 x)^2} dx = -\frac{\partial J}{\partial b} = \frac{\pi}{4\sqrt{ab^3}}.$$

Adding the two results then produces

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{(a \cos^2 x + b \sin^2 x)^2} dx = \frac{\pi}{4\sqrt{ab}} \left(\frac{1}{a} + \frac{1}{b} \right),$$

which is the value of the integral I .

Symbolic algorithms

Many problems in mathematics, physics, and engineering involve integration where an explicit formula for the integral is desired. Extensive tables of integrals have been compiled and published over the years for this purpose. With the spread of computers, many professionals, educators, and students have turned to computer algebra systems that are specifically designed to perform difficult or tedious tasks, including integration. Symbolic integration presents a special challenge in the development of such systems.

A major mathematical difficulty in symbolic integration is that in many cases, a closed formula for the antiderivative of a rather simple-looking function does not exist. For instance, it is known from Liouville's Theorem that the antiderivatives of the functions $\exp(x^2)$, x^x and $(\sin x)/x$ cannot be expressed in the closed form involving only elementary functions: rational and exponential functions, logarithm, trigonometric and inverse trigonometric functions, and the operations of multiplication and composition. *Differential Galois theory* provides general criteria that allow one to determine whether the antiderivative of an elementary function is elementary. Unfortunately, it turns out that functions with closed expressions of antiderivatives are the exception rather than the rule. Consequently, computerized algebra systems have no hope of being able to find an antiderivative for a randomly constructed elementary function. On the positive side,

if the ‘building blocks’ for antiderivatives are fixed in advance, it may be still be possible to decide whether the antiderivative of a given function can be expressed using these blocks and operations of multiplication and composition, and to find the symbolic answer whenever it exists. The Risch algorithm, implemented in *Maple*, *Mathematica* and other computer algebra systems, does just that for functions and antiderivatives built from rational functions, radicals, logarithm, and exponential functions.

Some special integrands occur often enough to warrant special study. In particular, it may be useful to have, in the set of antiderivatives, the special functions like the Legendre functions, the hypergeometric function, the Gamma function, the Lambert-W function and so on. Extending the Risch-Norman algorithm so that it includes these functions is possible but challenging.

Numerical Integration

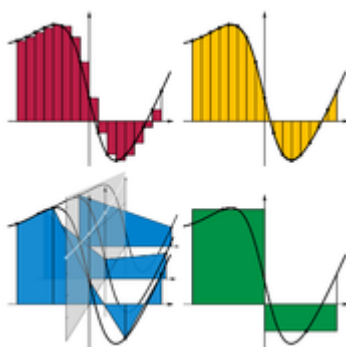
The integrals encountered in a basic calculus course are deliberately chosen for simplicity; those found in real applications are not always so accommodating. Some integrals cannot be found exactly, some require special functions which themselves are a challenge to compute, and others are so complex that finding the exact answer is too slow. This motivates the study and application of numerical methods for approximating integrals, which today use floating point arithmetic on digital electronic computers.

The goals of numerical integration are accuracy, reliability, efficiency, and generality. Sophisticated methods can vastly outperform a naive method by all four measures. Consider, for example, the integral

$$\int_{-2}^2 \frac{1}{5} \left(\frac{1}{100} (322 + 3x(98 + x(37 + x))) - 24 \frac{x}{1+x^2} \right) dx = \frac{94}{25} = 3.76.$$

The normal calculus approach divides the integration range into, say, 16 equal pieces, and computes function values.

Spaced function values									
x	-2.00	-1.50	-1.00	-0.50	0.00	0.50	1.00	1.50	2.00
f(x)	2.22800	2.45663	2.67200	2.32475	0.64400	-0.92575	-0.94000	-0.16963	0.83600
x		-1.75	-1.25	-0.75	-0.25	0.25	0.75	1.25	1.75
f(x)		2.33041	2.58562	2.62934	1.64019	-0.32444	-1.09159	-0.60387	0.31734



Numerical quadrature methods: ■ Rectangle, ■ Trapezoid, ■ Romberg, ■ Gauss

Using the left end of each piece, the rectangle method sums 16 function values and multiplies by the step width, h , here 0.25, to get an approximate value of 3.94325 for the integral. The accuracy is not impressive, but calculus formally uses pieces of infinitesimal width, so initially this may seem little cause for concern. Indeed, repeatedly doubling the number of steps eventually produces an approximation of 3.76001. However 2^{18} pieces are required, a great computational expense for so little accuracy; and a reach for greater accuracy can force steps so small that arithmetic precision becomes an obstacle. Note that there is a Left Sum (LSUM) and a Right Sum (RSUM).

A better approach replaces the horizontal tops of the rectangles with slanted tops touching the function at the ends of each piece. The Trapezoid Rule is almost as easy to calculate; it sums all 17 function values, but weights the first and last by one half, and again multiplies by the step width. This immediately improves the approximation to 3.76925, which is noticeably more accurate. Furthermore, only 2^{10} pieces are needed to achieve 3.76000, substantially less computation than the rectangle method for comparable accuracy. Note here that $\text{TRAP} = (\text{LSUM} + \text{RSUM})/2$

Romberg's method builds on the trapezoid method to great effect. First, the step lengths are halved incrementally, giving trapezoid approximations denoted by $T(h_0)$, $T(h_1)$, and so on, where h_{k+1} is half of h_k . For each new step size, only half the new function values need to be computed; the others carry over from the previous size (as shown in the table above). But the really powerful idea is to interpolate a polynomial through the approximations, and extrapolate to $T(0)$. With this method a numerically *exact* answer here requires only four pieces (five function values)! The Lagrange polynomial interpolating $\{h_k, T(h_k)\}_{k=0..2} = \{(4.00, 6.128), (2.00, 4.352), (1.00, 3.908)\}$ is $3.76 + 0.148h^2$, producing the extrapolated value 3.76 at $h = 0$.

Gaussian quadrature often requires noticeably less work for superior accuracy. In this example, it can compute the function values at just two x positions, $\pm\sqrt{3}$, then double each value and sum to get the numerically exact answer. The explanation for this dramatic success lies in error analysis, and a little luck. An n -point Gaussian method is exact for polynomials of degree up to $2n-1$. The function in this example is a degree 3 polynomial, plus a term that cancels because the chosen endpoints are symmetric around zero. (Cancellation also benefits the Romberg method.)

Shifting the range left a little, so the integral is from -2.25 to 1.75 , removes the symmetry. Nevertheless, the trapezoid method is rather slow, the polynomial interpolation method of Romberg is acceptable, and the Gaussian method requires the least work — if the number of points is known in advance. As well, rational interpolation can use the same trapezoid evaluations as the Romberg method to greater effect.

Quadrature method cost comparison				
Method	Trapezoid	Romberg	Rational	Gauss
Points	1048577	257	129	36
Rel. Err.	-5.3×10^{-13}	-6.3×10^{-15}	8.8×10^{-15}	3.1×10^{-15}
Value	$\int_{-2.25}^{1.75} f(x) dx = 4.1639019006585897075 \dots$			

In practice, each method must use extra evaluations to ensure an error bound on an unknown function; this tends to offset some of the advantage of the pure Gaussian method, and motivates the popular Gauss–Kronrod quadrature formulas. Symmetry can still be exploited by splitting this integral into two ranges, from -2.25 to -1.75 (no symmetry), and from -1.75 to 1.75 (symmetry). More broadly, adaptive quadrature partitions a range into pieces based on function properties, so that data points are concentrated where they are needed most.

A calculus text is no substitute for numerical analysis, but the reverse is also true. Even the best adaptive numerical code sometimes requires a user to help with the more demanding integrals. For example, improper integrals may require a change of variable or methods that can avoid infinite function values; and known properties like symmetry and periodicity may provide critical leverage.

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External links

- [The Integrator](#) by [Wolfram Research](#)
- [Riemann Sum](#) by [Wolfram Research](#)
- [Mathematical Assistant on Web](#) online calculation of integrals, allows to integrate in small steps (includes also hints for next step which cover techniques like by parts, substitution, partial fractions, application of formulas and others, powered by [Maxima \(software\)](#))
- P.S. Wang, [Evaluation of Definite Integrals by Symbolic Manipulation](#) (1972) - a cookbook of definite integral techniques