MATH 6102
Spring 2009

Differential Equations

What is a Differential Equation?

- Derivative measures rate of change at a point
- Derivative measures slope of tangent line to function at the point
- If we knew rate of change at each point can we reconstruct the original function?
- Yes – up to a constant! Antidifferentiation
- This leads to an equation

$$\frac{dy}{dx} = F(x, y)$$

What is a Differential Equation?

A **differential equation** is an equation involving an unknown function and its derivatives. The **order** of the differential equation is the order of the highest derivative of the unknown function involved in the equation. A **linear differential equation** of order *n* is a differential equation written in the following form:

$$e_n(x)\frac{d^n y}{dx^n} + e_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + e_1(x)\frac{dy}{dx} + e_0(x)y = f(x)$$

Where $e_n(x)$ is not the zero function.

Examples

Exponential:

$$\frac{dy}{dt} = ky$$

$$y' = ky$$

Newton's Law of Cooling: the *rate of change of the temperature difference between an object and its surroundings is proportional to the temperature difference itself*

$$\frac{dy}{dt} = k(M - y)$$

Examples

Logistic: $y(t)$ = population density

r = reproductive rate

K = carrying capacity

(1) an equation in which the rate of reproduction is no longer constant but declines with the density of the population

(2) if we expand out the equation suggests a process in which the birth rate is still constant , but in which there is a a mortality term, that dominates when the population is high

$$\frac{dy}{dt} = ry\frac{(K-y)}{K} = ry - \frac{r}{K}y^2$$

Solution to a Differential Equations

A solution to a differential equation is a function defined on the domain of the equation that makes the equation true.

IVP (***initial value problem***) is a differential equation together with a specific value that the solution must satisfy at a particular point.

If no initial conditions are given, we call the description of all solutions to the differential equation the **general solution**.

When do solutions exist?

Theorem. *Let $f(x,y)$ be a real valued function which is continuous on the rectangle $R = \{(x,y) \mid |x - x_o| < a, |y - y_o| < b\}$. Assume f has a partial derivative with respect to y which is continuous on the rectangle R. Then there exists an interval I (with $I = [x_o - h, x_o + h]$, $h \leq a$) such that the initial value problem*

$$y'(x) = f(x,y), \quad y(x_o) = y_o$$

has a unique solution $y(x)$ defined on the interval I.

Picard's Method of Successive Approximations

$$y'(x) = f(x,y), \quad y(x_o) = y_o$$

Step 1. Consider the constant function

$$y(x) = y_o$$

This clearly satisfies the initial condition. It may not satisfy the differential equation though.

Step 2. Once the function $y_n(x)$ is known, define the function

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))\, dt$$

This function satisfies the initial condition and is closer in value solving the DE.

Picard's Method

Step 3. By induction, we generate a sequence of functions $\{y_n(x)\}$ which, under the assumptions made on , converges to the solution $y(x)$ of the initial value problem.

Picard's Method

Example 1:

Consider the differential equation:

$$y'(x) = x + y(x),\ y(0) = 0.$$

Step 1: Set $y_0(x) = 0$. Then this satisfies the IVP.

Step 2: New solution –

$$y_1(x) = y_0 + \int_0^x f(t, y_0(x))dt$$
$$= 0 + \int_0^x t\, dt$$
$$= x^2/2$$

$$y_2(x) = y_0 + \int_0^x f(t, y_1(t))dt = 0 + \int_0^x \left(t + \frac{t^2}{2}\right)dt$$

$$= \frac{1}{2}x^2 + \frac{1}{6}x^3$$

Picard's Method

$$y_3(x) = y_0 + \int_0^x f(t, y_2(t))\,dt = 0 + \int_0^x \left(t + \frac{t^2}{2} + \frac{t^3}{6} \right) dt$$

$$= \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$$

$$y_4(x) = y_0 + \int_0^x f(t, y_3(t))\,dt$$

$$= \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

$$y_n(x) = \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^{n+1}}{(n+1)!}$$

Picard's Method

$$y_n(x) = \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^{n+1}}{(n+1)!}$$

$$= -1 - x + \sum_{k=0}^{n+1} \frac{x^k}{k!}$$

$$y(x) = \lim_{n \to 0} \left(-1 - x + \sum_{k=0}^{n+1} \frac{x^k}{k!} \right)$$

$$= -1 - x + \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$y(x) = -1 - x + e^x$$

Picard's Method

Example:

Consider the differential equation:

$$y'(x) = x + (y(x))^2, \; y(0) = 0.$$

Step 1: Set $y_0(x) = 0$. Then this satisfies the IVP.

Step 2: New solution –

$$y_1(x) = y_0 + \int_0^x f(t, y_0(x))dt$$
$$= 0 + \int_0^x t\, dt$$
$$= x^2/2$$

$$y_2(x) = y_0 + \int_0^x f(t, y_1(t))dt = 0 + \int_0^x \left(t + \left(\frac{t^2}{2} \right)^2 \right) dt$$

$$= \frac{1}{2}x^2 + \frac{1}{20}x^5$$

Picard's Method

$$y_3(x) = y_0 + \int_0^x f(t, y_2(t))\,dt = 0 + \int_0^x \left(t + \left(\frac{t^2}{2} + \frac{t^5}{20} \right)^2 \right) dt$$

$$= \frac{1}{2}x^2 + \frac{1}{20}x^5 + \frac{1}{160}x^8 + \frac{1}{4400}x^{11}$$

$$y_4(x) = y_0 + \int_0^x f(t, y_3(t))\,dt$$

$$= \frac{x^2}{2} + \frac{x^5}{20} + \frac{x^8}{160} + \frac{x^{11}}{4{,}400} + \frac{3x^{14}}{49{,}280}$$

$$+ \frac{87x^{17}}{23{,}936{,}000} + \frac{x^{20}}{7{,}040{,}000} + \frac{x^{11}}{445{,}280{,}000}$$

Picard's Method

$$y(x) = \lim_{n\to\infty} y_n(x) = ?$$

$$= \frac{\sqrt{3}AiryAi(1,-x) + AiryBi(1,-x)}{\sqrt{3}AiryAi(-x) + AiryBi(-x)} \qquad \text{(according to } \textit{Maple}\text{)}$$

$$= \frac{x^2 I_{2/3}\left(\frac{2ix^{3/2}}{3}\right)}{ix^{3/2} I_{-1/3}\left(\frac{2ix^{3/2}}{3}\right)}, \quad \textit{where } I_\alpha(x) = i^{-\alpha} J_\alpha(ix) \textit{ and}$$

$$J_\alpha(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}$$

Even though we have a solution, it is not very helpful!

Picard's Method

Use Picard's Method to find a solution to

$$\frac{dy}{dx} = x^2 y, \quad y(0) = 1$$

Picard's Method

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$$\frac{dy}{dx} = x^2 y, \quad y(0) = 1$$

$$y_1(x) = y_0 + \int_0^x t^2 y_0 \, dt = 1 + \frac{1}{3}x^3$$

$$y_2(x) = y_0 + \int_0^x t^2 y_1 \, dt = 1 + \frac{1}{3}x^3 + \frac{1}{18}x^6$$

$$y_3(x) = y_0 + \int_0^x t^2 y_2 \, dt = 1 + \frac{1}{3}x^3 + \frac{1}{2 \cdot 3^2}x^6 + \frac{1}{2 \cdot 3^4}x^9$$

$$y_4(x) = y_0 + \int_0^x t^2 y_3 \, dt = 1 + \frac{1}{3}x^3 + \frac{1}{2! \cdot 3^2}x^6 + \frac{1}{3! \cdot 3^3}x^9 + \frac{1}{4! \cdot 3^4}x^{12}$$

$$\vdots$$

$$y_n(x) = \sum_{k=0}^{n} \frac{1}{k! 3^k} x^{3k} = \sum_{k=0}^{n} \frac{1}{k!} \left(\frac{x^3}{3} \right)^k$$

Picard's Method

Use Picard's Method to find a solution to

$$\frac{dy}{dx} = x^2 y, \quad y(0) = 1$$

$$y_n(x) = \sum_{k=0}^{n} \frac{1}{k!3^k} x^{3k} = \sum_{k=0}^{n} \frac{1}{k!}\left(\frac{x^3}{3}\right)^k$$

$$y(x) = \lim_{n\to\infty} y_n(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k!}\left(\frac{x^3}{3}\right)^k = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{x^3}{3}\right)^n$$

$$y(x) = e^{x^3/3}$$

Solution by Series

If we know that a solution exists, it is clearly differentiable since it satisfies a differential equation. We can assume that it is analytic and approximate it by a power series:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Now, use this for the function in the differential equation and look for relationships among the coefficients.

Solution by Series

Consider the differential equation:

$$y''(x) - y(x) = 0.$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$y''(x) - y(x) = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$(2a_2 - a_0) + (6a_3 - a_1)x + (12a_4 - a_2)x^2 + \ldots = 0$$

Solution by Series

$$\left.\begin{aligned} 2a_2 - a_0 &= 0 \\ 6a_3 - a_1 &= 0 \\ 12a_4 - a_2 &= 0 \\ &\vdots \\ n(n-1)a_n - a_{n-2} &= 0 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} &a_2 = a_0/2! \\ &a_3 = a_1/3! \\ &a_4 = a_2/12 = a_0/4! \\ &\quad\vdots \\ &a_n = a_\beta / n! \begin{cases} \beta = 0 \text{ if } n \text{ is even} \\ \beta = 1 \text{ if } n \text{ is odd} \end{cases} \end{aligned}\right.$$

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

$$y(x) = a_0 \cosh(x) + a_1 \sinh(x)$$

Solution by Series

Let's return to the differential equation:

$$y'(x) = x + (y(x))^2.$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\left[y(x)\right]^2 = a_0{}^2 + 2a_0 a_1 x + \left(2a_0 a_2 + a_1{}^2\right)x^2 + \left(2a_0 a_3 + 2a_1 a_2\right)x^3$$
$$+\left(2a_0 a_4 + 2a_1 a_3 + a_2{}^2\right)x^4 + \left(2a_0 a_5 + 2a_1 a_4 + a_2 a_3\right)x^5 + \ldots$$

$$y'(x) = x + \left(y(x)\right)^2$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = x + a_0{}^2 + 2a_0 a_1 x + \left(2a_0 a_2 + a_1{}^2\right)x^2 + \left(2a_0 a_3 + 2a_1 a_2\right)x^3 + \ldots$$

Solution by Series

$$\sum_{n=1}^{\infty} n a_n x^{n-1} - x - \left(a_0{}^2 - 2a_0 a_1 x - \left(2a_0 a_2 + a_1{}^2\right)x^2 - \left(2a_0 a_3 + 2a_1 a_2\right)x^3 - \ldots\right) = 0$$

$$\left.\begin{aligned} a_1 - a_0{}^2 &= 0 \\ 2a_2 - 1 - 2a_0 a_1 &= 0 \\ 3a_3 - \left(2a_0 a_2 + a_1{}^2\right) &= 0 \\ &\vdots \end{aligned}\right\} \Rightarrow \begin{cases} a_1 = a_0{}^2 \\ a_2 = \dfrac{1 + 2a_0 a_1}{2} = \dfrac{1}{2} + a_0{}^3 \\ a_3 = \dfrac{2a_0 a_2 + a_1{}^2}{3} = \dfrac{1}{3} a_0 + a_0{}^4 \\ a_4 = \dfrac{2a_0 a_3 + 2a_1 a_2}{4} = \dfrac{7}{24} a_0{}^2 + \dfrac{5}{8} a_0{}^5 \\ a_5 = \dfrac{2a_0 a_4 + 2a_1 a_3 + a_2^2}{4} = \dfrac{1}{20} + \dfrac{27}{60} a_0{}^3 + \dfrac{17}{20} a_0{}^6 \\ \quad \vdots \end{cases}$$

Solution by Series

Use series to find a solution to the differential equation:

$$y'(x) = 3y(x) + 6, \;\; y(0)=7$$

Solution by Series

Use series to find a solution to the differential equation:

$$y'(x) = 3y(x) + 6, \quad y(0)=7$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'(x) = 3y(x) + 6$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = 6 + \sum_{n=0}^{\infty} 3 a_n x^n$$

$$a_1 = 6 + 3a_0$$

$$2a_2 = 3a_1$$

$$3a_3 = 3a_2$$

$$\vdots$$

Solution by Series

$$\left.\begin{aligned} a_1 &= 6 + 3a_0 \\ 2a_2 &= 3a_1 \\ 3a_3 &= 3a_2 \\ &\vdots \\ na_n &= 3a_{n-1} \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} a_1 &= 6 + 3a_0 \\ a_2 &= \frac{18 + 9a_0}{2} \\ a_3 &= \frac{54 + 27a_0}{6} \\ a_4 &= \frac{3^4(2 + a_0)}{4!} \\ &\vdots \\ a_n &= \frac{3^n(2 + a_0)}{n!} \end{aligned}\right.$$

Solution by Series

$$y(x) = a_0 + \sum_{n=1}^{\infty} \frac{3^n(2+a_0)}{n!} x^n$$

$$= -2 + (2+a_0) + \sum_{n=1}^{\infty} \frac{3^n(2+a_0)}{n!} x^n$$

$$= -2 + (2+a_0) \sum_{n=0}^{\infty} \frac{(3x)^n}{n!}$$

$$= -2 + (2+a_0) e^{3x}$$

Since $y(0)=7$, $a_0 = 7$ and

$$y(x) = -2 + 9e^{3x}$$

Are there other ways to solve DE's?

There are multiple techniques for solving differential equations but that is for a DE course. The idea here is to see how these objects are related to what we have been studying, thus the power series solutions.

We will look at one type of DE that is easy to solve.

The other way of thinking about differential equations is geometrically. The derivative measures the slope of the tangent line. How can we use this?

Separable Differential Equations

We have a differential equation of the form.

$$\frac{dy}{dx} = F(x,y) = f(x)g(y)$$

Assume that the right hand side of the first order DE can be written as a product of a function of x and a function of y. If this is true, the equation is said to be *separable*.

If this is the case, there is a chance that the equation can be solved by antiderivatives.

Separable Differential Equations

$$\frac{dy}{dx} = f(x)g(y)$$

$$\frac{dy}{g(y)} = f(x)dx$$

$$\int \frac{dy}{g(y)} = \int f(x)dx$$

If both antiderivatives can be found and if the antiderivative of the left is invertible, there is an explicit solution for y(x). Otherwise, you may only have an implicit relationship.

Separable Differential Equations

$$\frac{dy}{dx} = x^2 y \qquad\qquad \frac{dy}{dx} = xy^2$$

$$\frac{dy}{y} = x^2\,dx \qquad\qquad \frac{dy}{y^2} = x\,dx$$

$$\int \frac{dy}{y} = \int x^2\,dx \qquad\qquad \int \frac{dy}{y^2} = \int x\,dx$$

$$\ln(y) = \frac{1}{3}x^3 + C \qquad\qquad -\frac{1}{y} = \frac{x^2}{2} + C$$

$$y(x) = e^{x^3/3+C} = Ae^{x^3/3} \qquad\qquad y(x) = -\frac{2}{x^2 + K}$$

Separable Differential Equations

Find the solution of the differential equation.

$$\frac{dy}{dx} = \frac{-xy}{\ln y}$$

that passes through the point $(0, e^2)$.

Separable Differential Equations

Find the solution of the differential equation.

$$\frac{dy}{dx} = -\frac{xy}{\ln y}$$

that passes through the point $(0, e^2)$.

$$\frac{dy}{dx} = -\frac{xy}{\ln y}$$

$$\int \frac{\ln y}{y}\, dy = \int -x\, dx$$

$$\frac{1}{2}(\ln y)^2 = -\frac{1}{2}x^2 + C$$

Separable Differential Equations

Now, $y(0) = e^2$.

$$\frac{1}{2}(\ln e^2)^2 = -\frac{1}{2}0^2 + C \Rightarrow C = 2$$

$$\frac{1}{2}(\ln y)^2 = -\frac{1}{2}x^2 + 2$$

$$\ln y = \sqrt{4 - x^2}$$

$$y = e^{\sqrt{4-x^2}}$$

The Logistic Equation

Find the solution of the differential equation.

$$\frac{dy}{dx} = -ky(M - y)$$

that passes through the point (0, A).

$$\frac{dy}{dx} = -ky(M - y)$$

$$\int \frac{dy}{y(M - y)} = \int -k\,dx$$

$$\frac{1}{M} \int \frac{1}{y} + \frac{1}{M - y} dy = -kx + C$$

$$\ln|y| - \ln|M - y| = -kMx + C_1$$

The Logistic Equation

Now, $y(0) = A$.

$$\ln\left|\frac{y}{M-y}\right| = -kMx + C_1$$

$$\ln\left|\frac{A}{M-A}\right| = C_1$$

$$\ln\left|\frac{y}{M-y}\right| = -kMx + \ln\left|\frac{A}{M-A}\right|$$

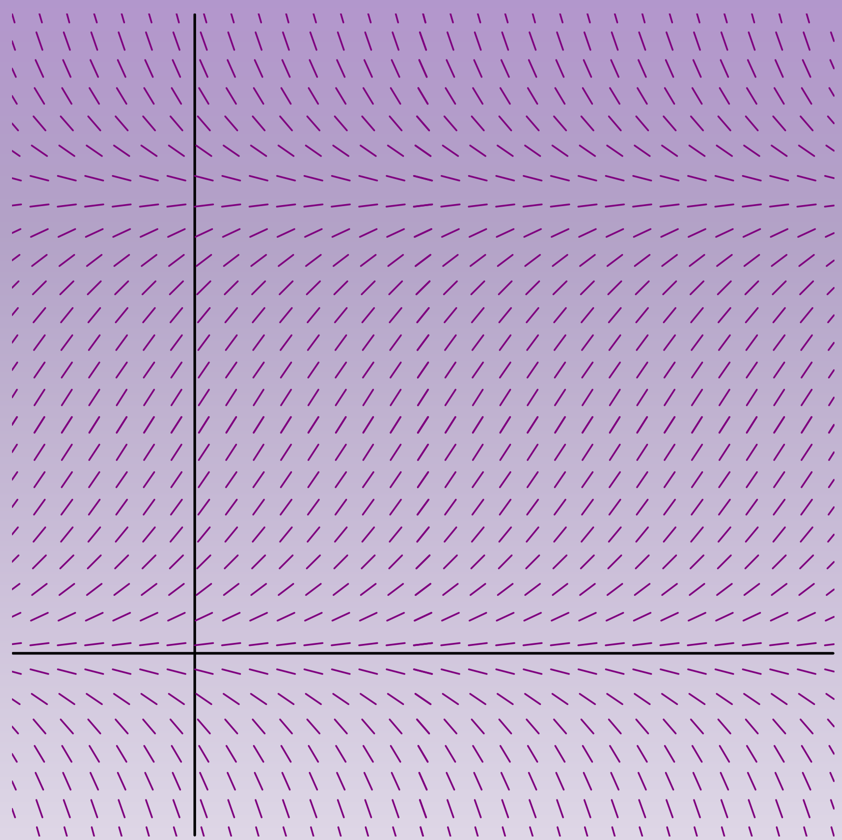
$$\frac{y}{M-y} = \frac{A}{M-A}e^{-kMx} = \beta e^{-kMx}$$

$$y = \frac{\beta M e^{-kMx}}{1+\beta e^{-kMx}}$$

Slope Fields

Consider this geometrically

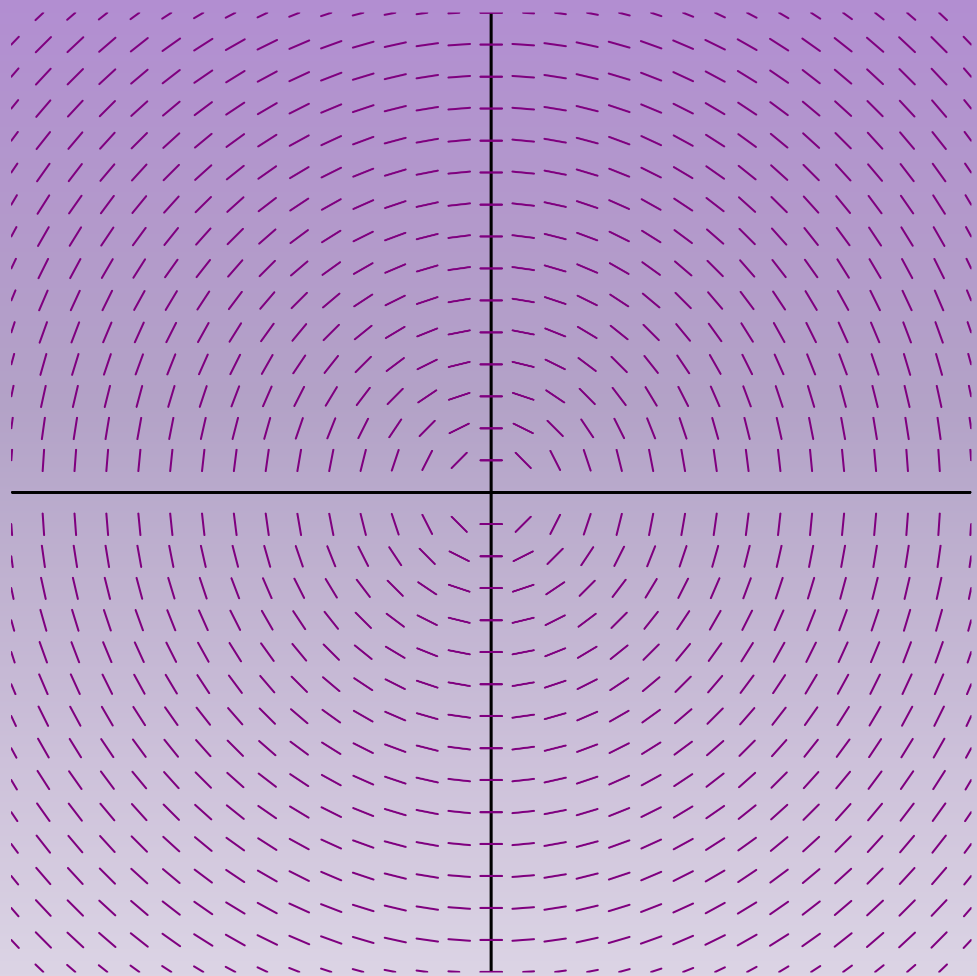
$$\frac{dy}{dx} = -ky(M-y)$$



Slope Fields

Consider the DE

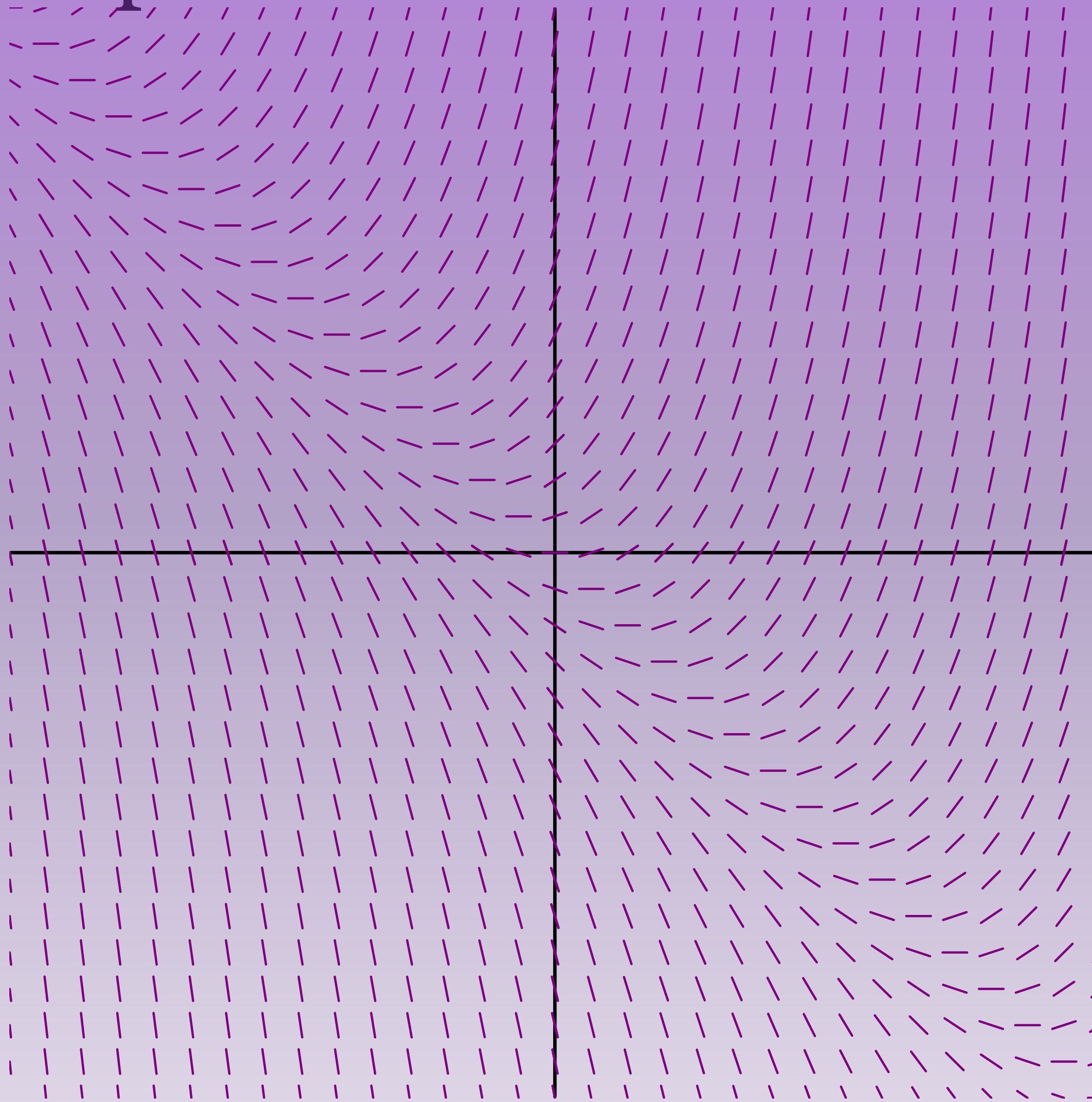
$$\frac{dy}{dx} = -\frac{x}{y}$$



Slope Fields

Consider the DE

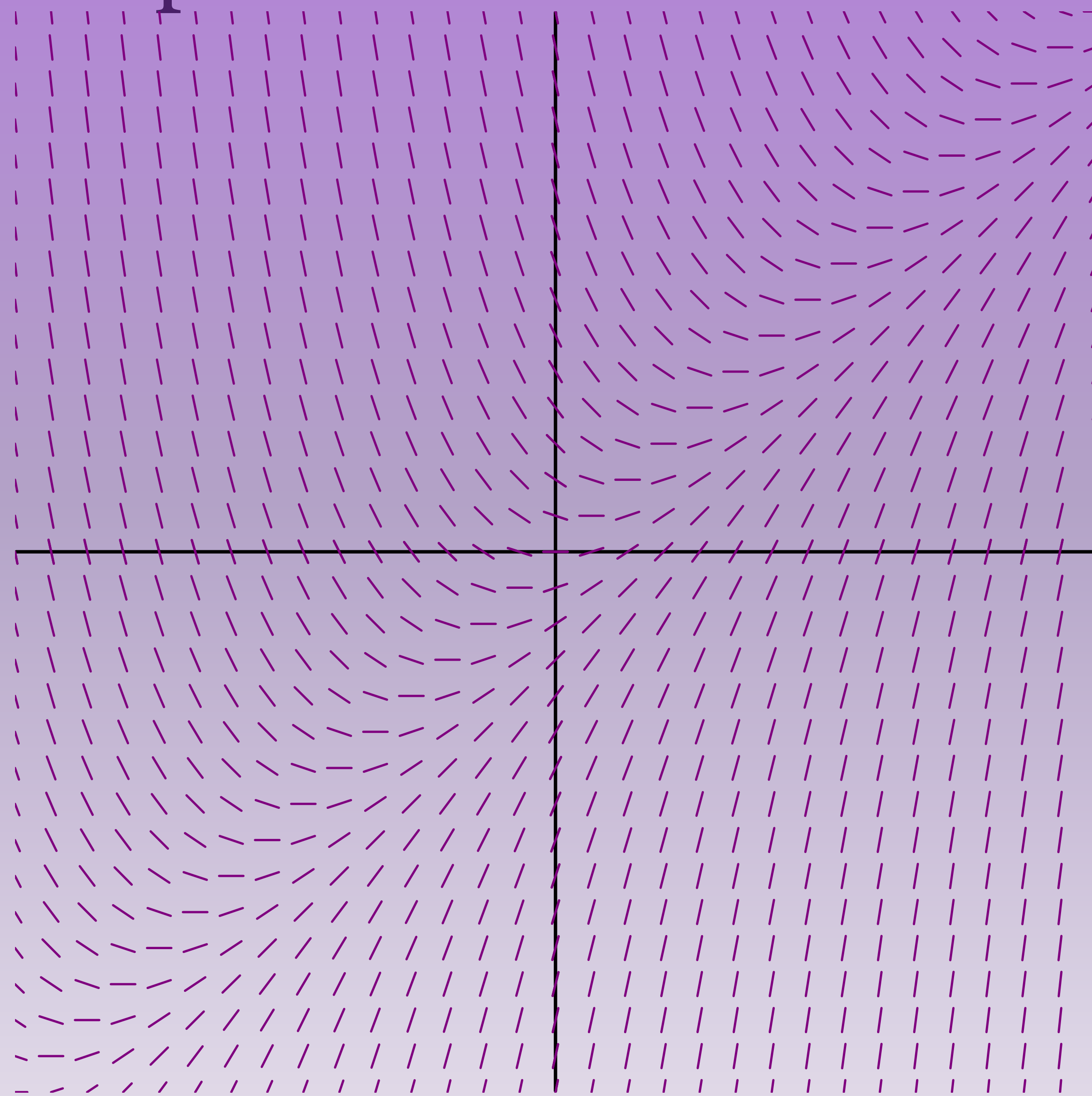
$$\frac{dy}{dx} = x + y$$



Slope Fields

Consider the DE

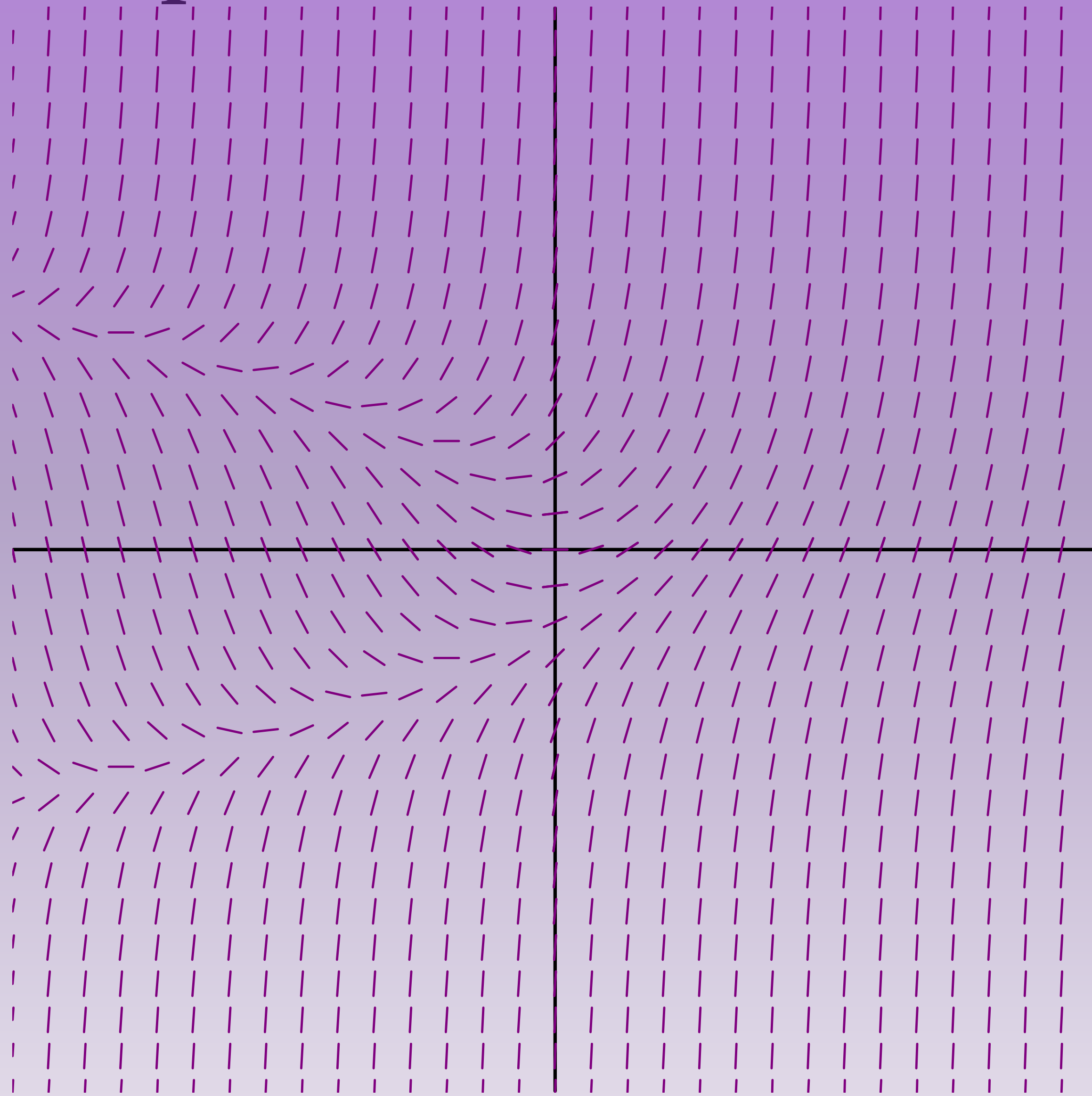
$$\frac{dy}{dx} = x - y$$



Slope Fields

Consider the DE

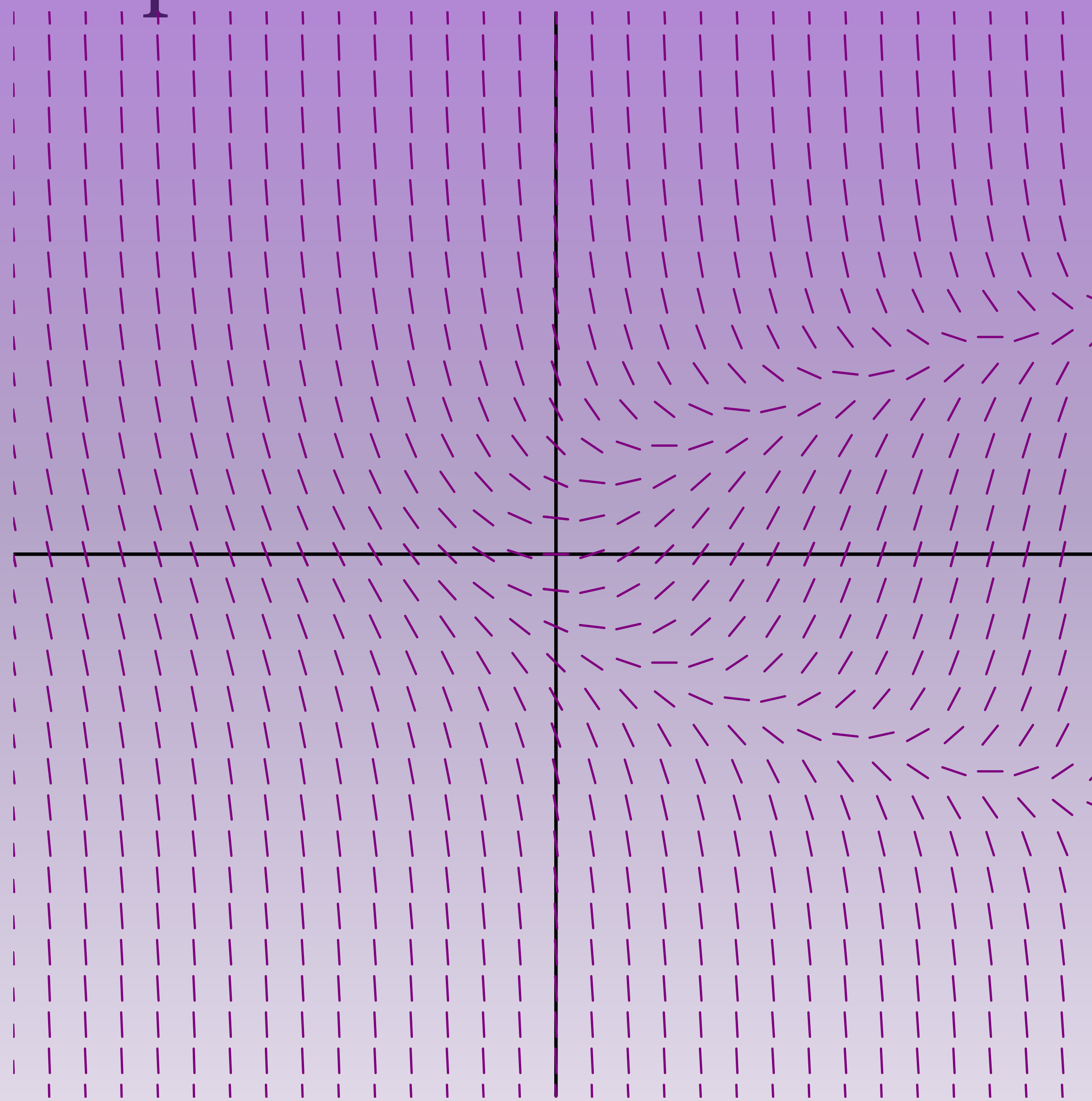
$$\frac{dy}{dx} = x + y^2$$



Slope Fields

Consider the DE

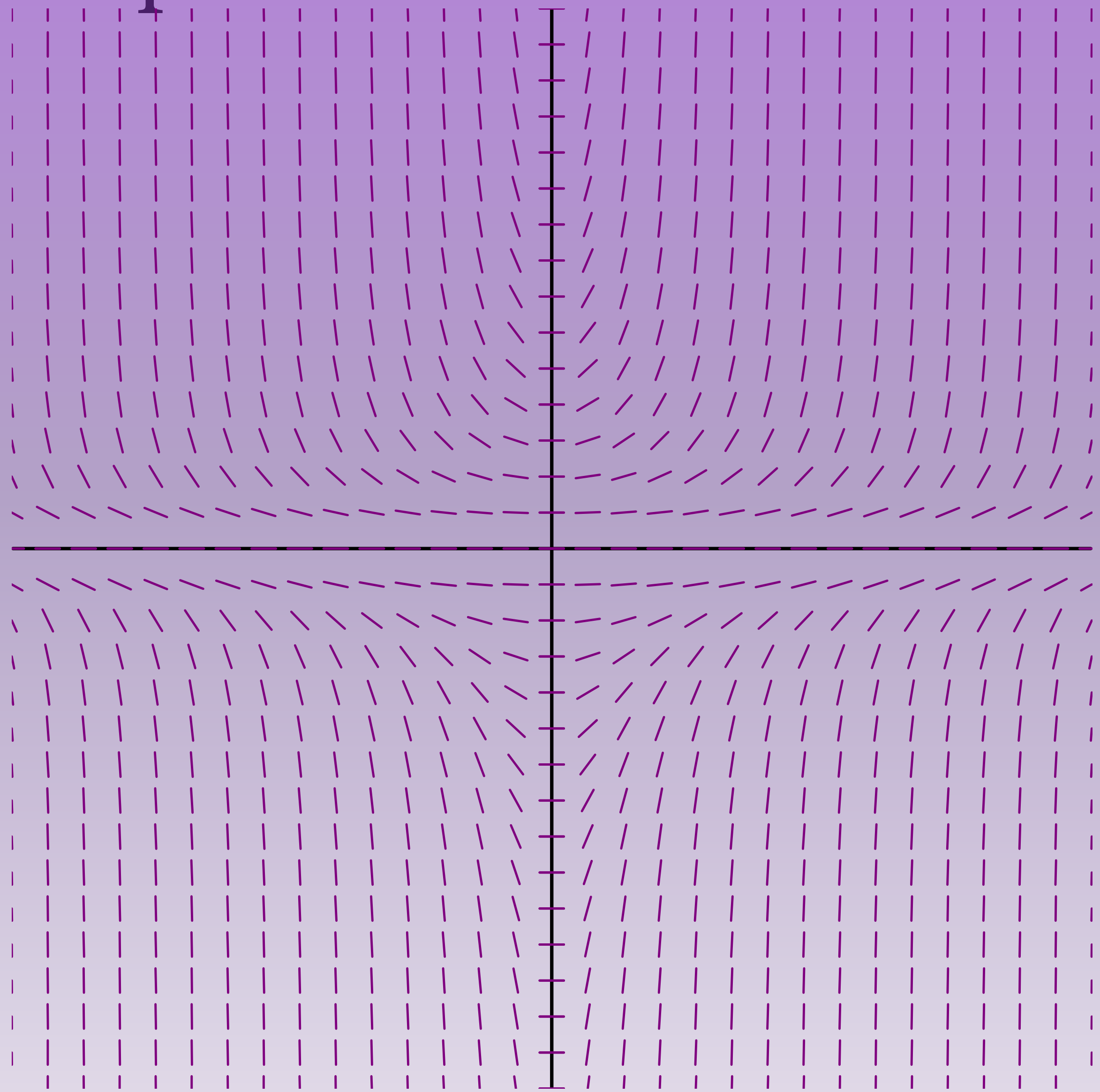
$$\frac{dy}{dx} = x - y^2$$



Slope Fields

Consider the DE

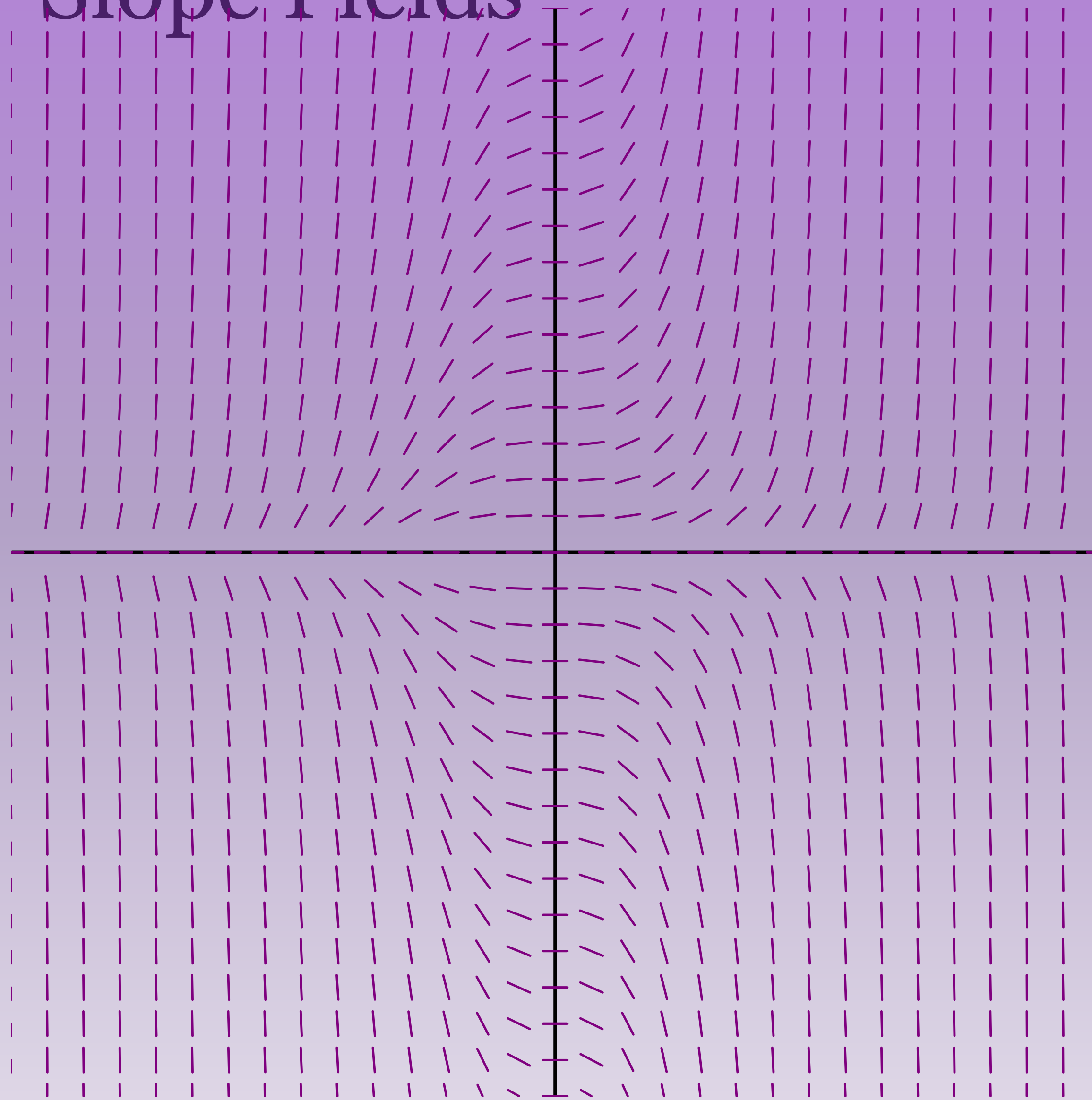
$$\frac{dy}{dx} = x^2 y$$



Slope Fields

Consider the DE

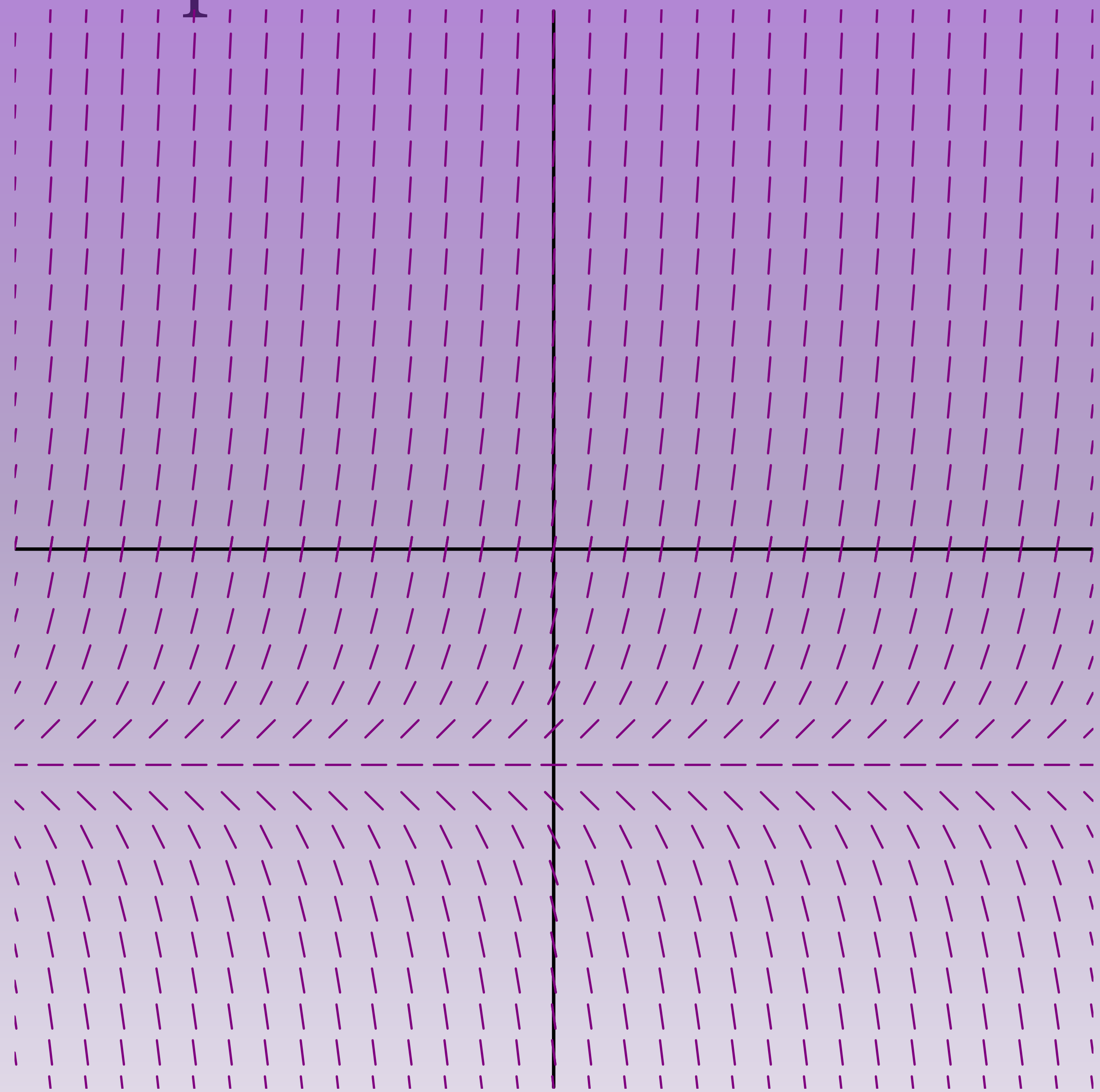
$$\frac{dy}{dx} = xy^2$$



Slope Fields

Consider the DE

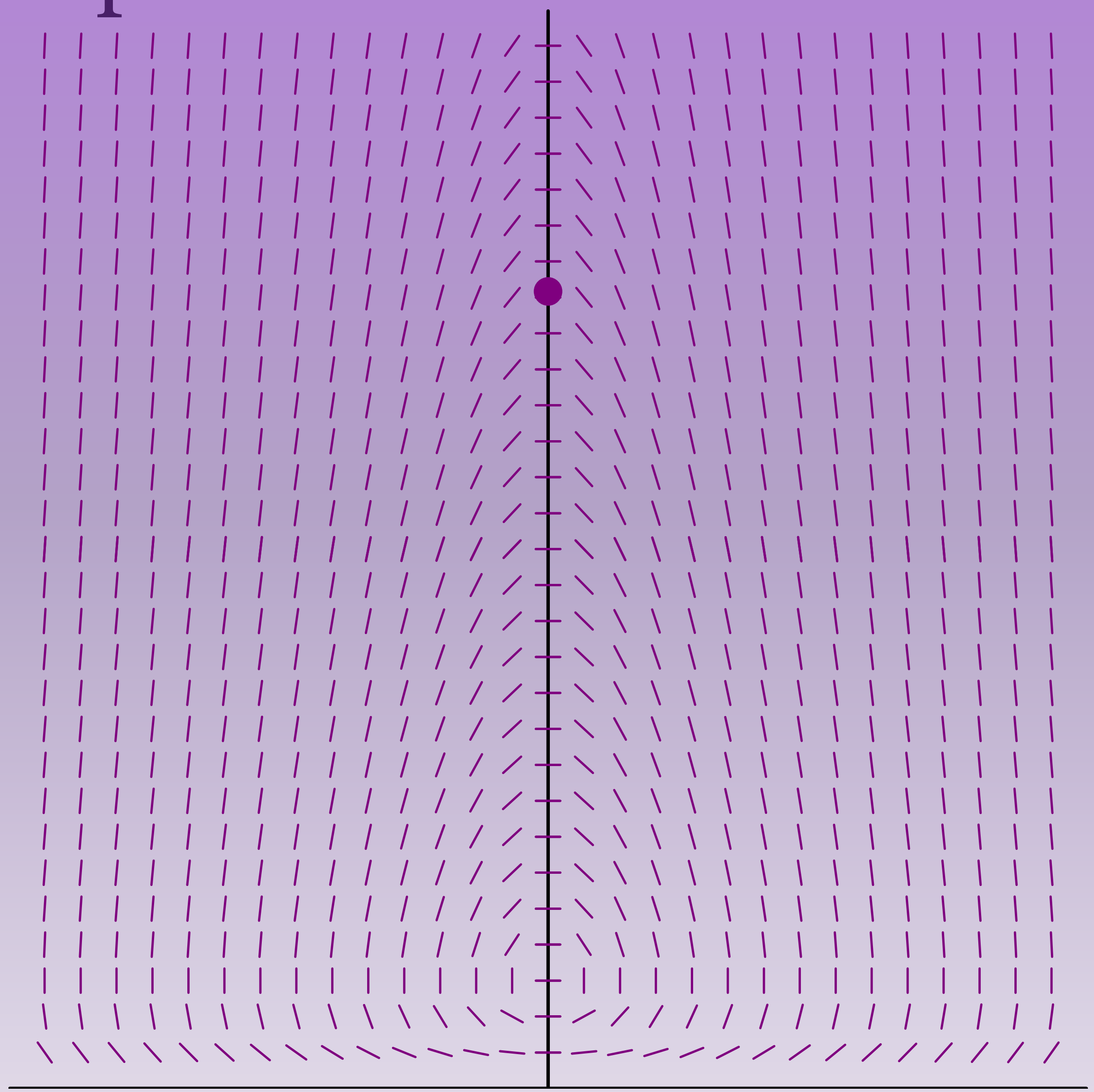
$$\frac{dy}{dx} = 3y + 6$$



Slope Fields

Consider the DE

$$\frac{dy}{dx} = -\frac{xy}{\ln y}$$

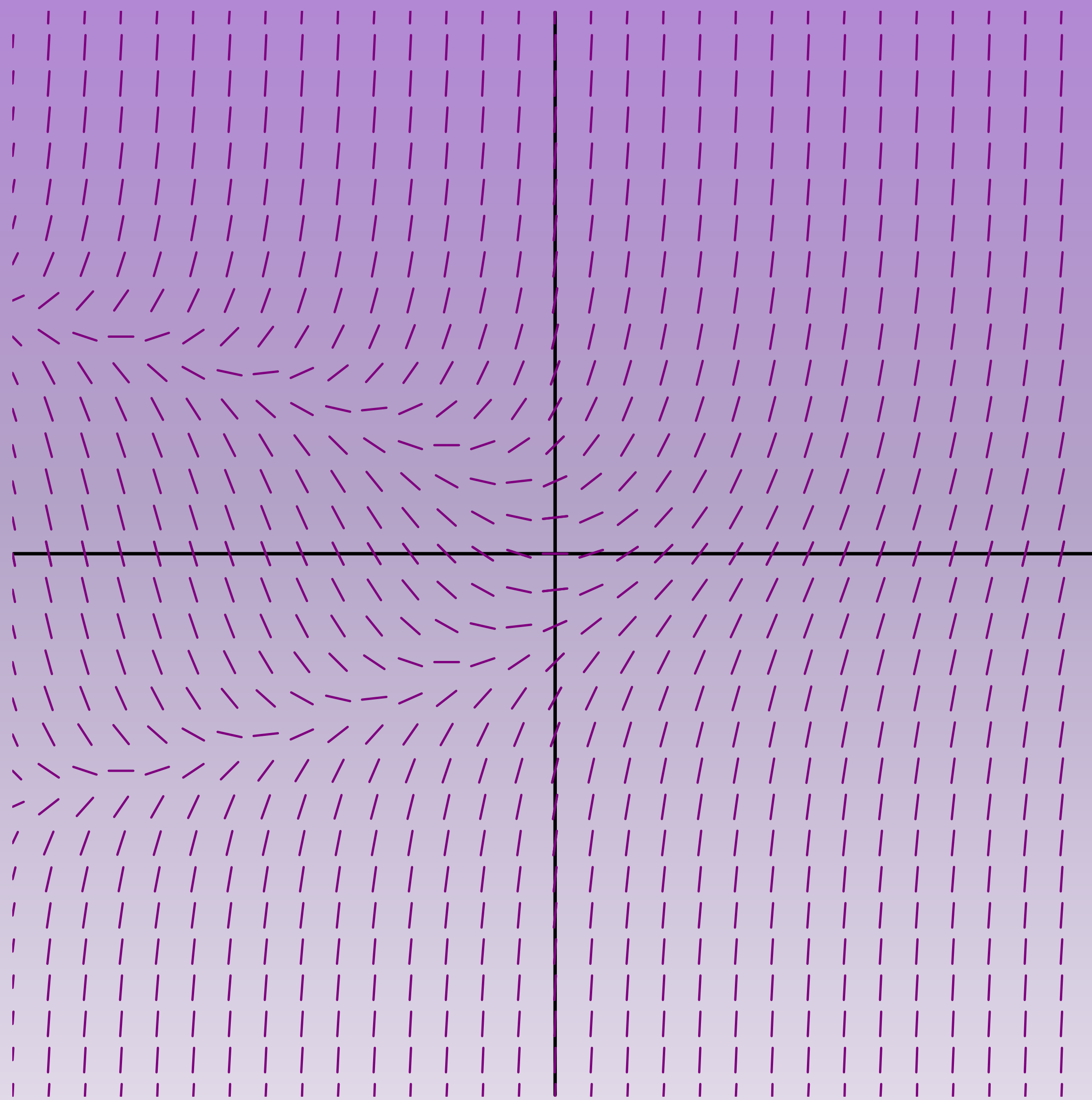


Euler's Method

Consider the DE graphically and numerically. If

$$\frac{dy}{dx} = x + y^2$$

Then starting at (0,0) we can move along the tangent line by a small step and see where we end up.



Euler's Method

$$\frac{dy}{dx} = x + y^2$$

At (0,0) the slope of the tangent line is $m = 0 + 0^2 = 0$. The equation of the tangent line is $y - 0 = m(x-0)$ or

$$y = 0.$$

Now, let x increase by 0.1. Where does that put us on the tangent line?

$$x_1 = x_0 + 0.1 = 0.1$$

$$y_1 = y(x_1) = 0$$

So $(x_1,y_1) = (0.1,0)$.

At (x_1,y_1) we need to compute the new slope:

$m_1 = (0.1) + 0^2 = 0.1$. The equation of the tangent line is

$$y - 0 = m(x - 0.1) = 0.1x - 0.01$$

Euler's Method

Now, let x increase by 0.1. Where does that put us on the tangent line?

$$x_2 = x_1 + 0.1 = 0.2$$

$$y_2 = y_1(x_2) = 0.01$$

So $(x_2,y_2) = (0.2,0.01)$.

At (x_2,y_2) we need to compute the new slope:

$m_2 = (0.2) + (0.01)^2 = 0.2001$. The equation of the tangent line is

$$y - 0.01 = m(x - 0.2) = 0.2001x - 0.04002$$

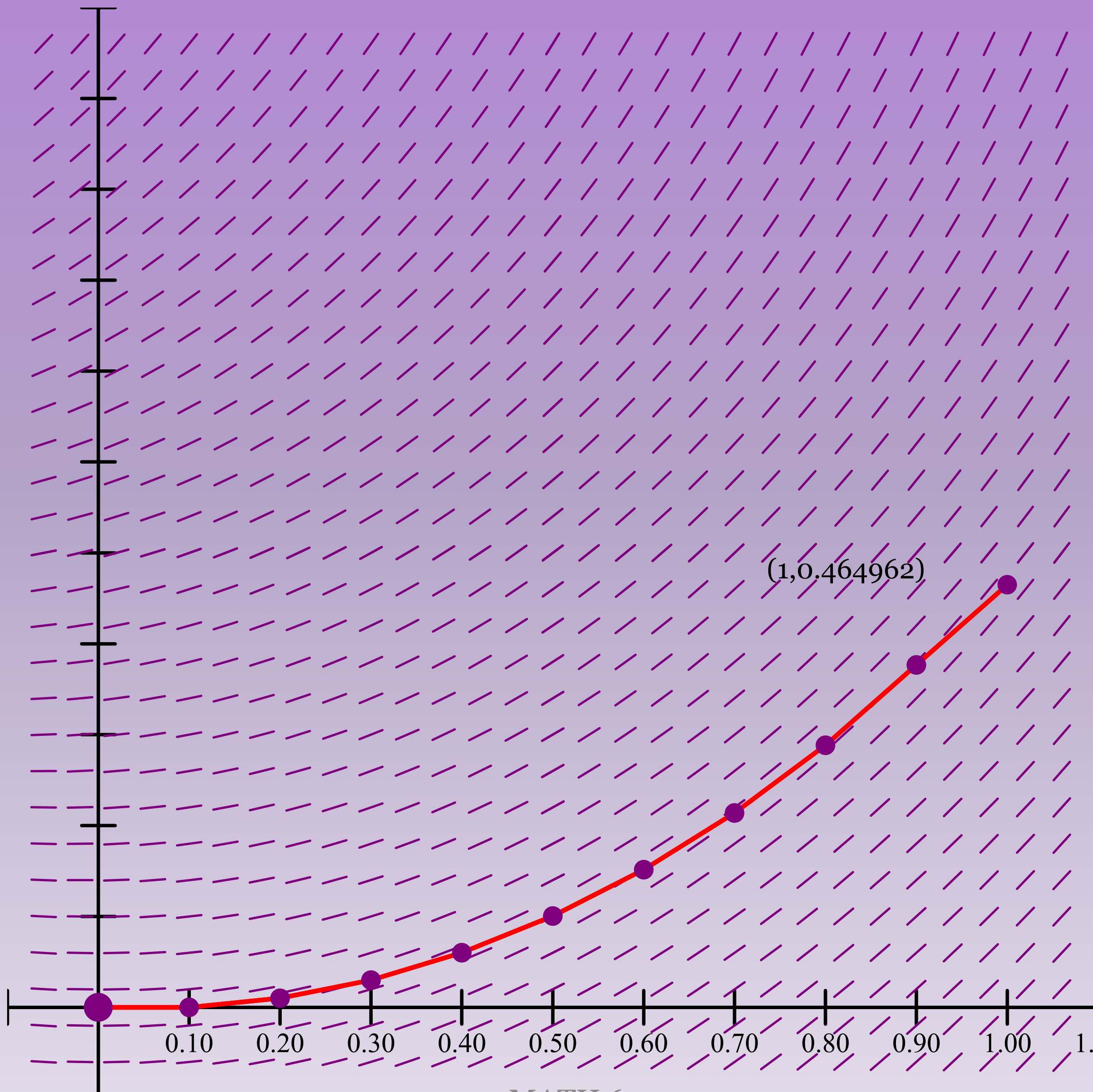
$$y = 0.2001x - 0.03002$$

We iterate this process until we get to the desired point.

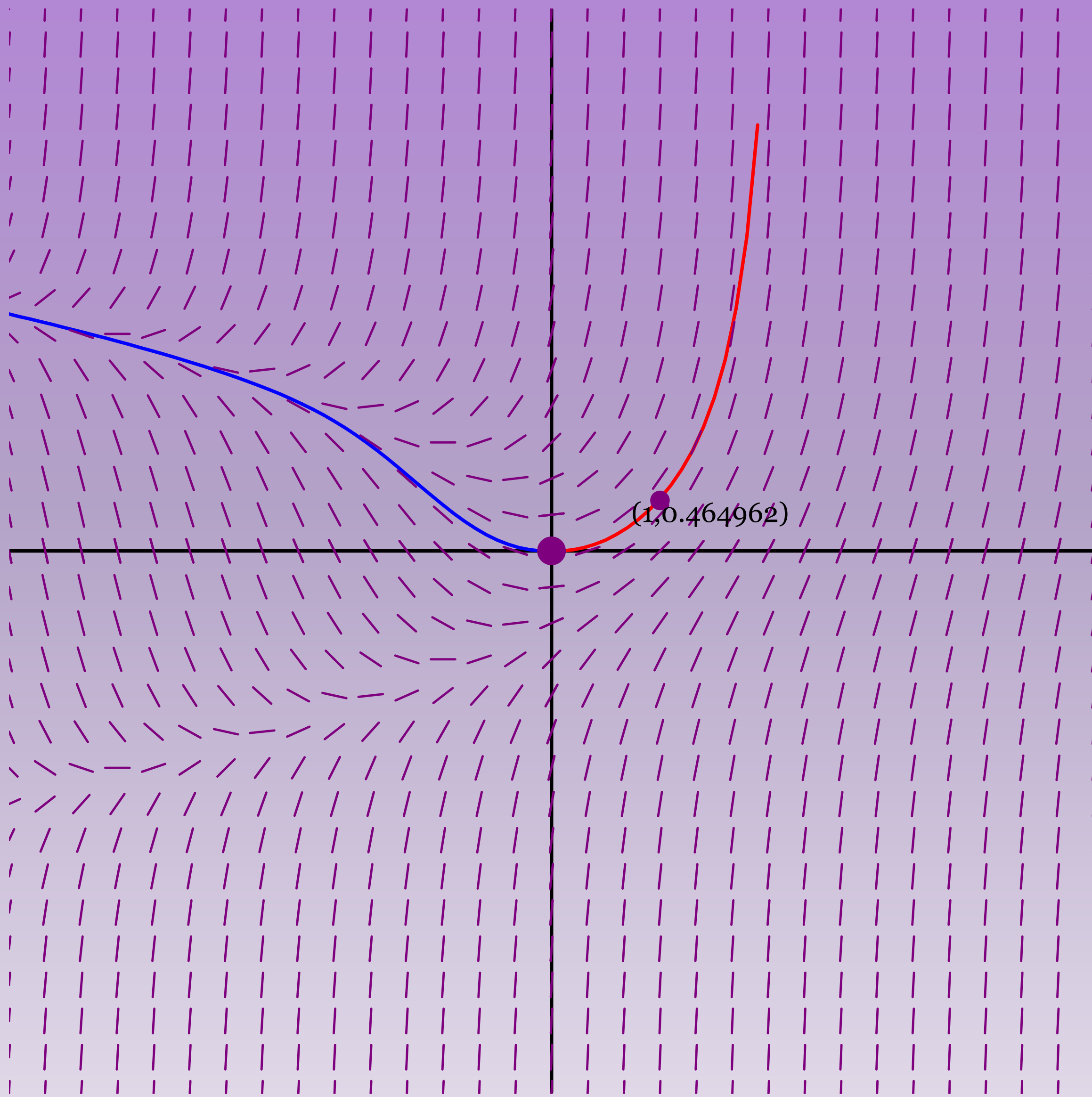
Euler's Method

n	$x_n = x_{old}$	$y_n = y_{old}$	m_n	Δx	Δy	x_{new}	y_{new}
0	0	0	0	0.1	0	0.1	0
1	0.1	0	0.1	0.1	0.01	0.2	0.01
2	0.2	0.01	0.2001	0.1	0.02001	0.3	0.03001
3	0.3	0.03001	0.300900	0.1	0.0300901	0.4	0.060100
4	0.4	0.06010	0.403612	0.1	0.0403612	0.5	0.100461
5	0.5	0.100461	0.510092	0.1	0.0510092	0.6	0.151470
6	0.6	0.151470	0.622943	0.1	0.0622943	0.7	0.213765
7	0.7	0.213765	0.745695	0.1	0.0745695	0.8	0.288334
8	0.8	0.288334	0.883136	0.1	0.0883136	0.9	0.376648
9	0.9	0.376648	1.041863	0.1	0.1041863	1.0	0.464962
10	1.0	0.464962		0.1			

Euler's Method



Euler's Method



Euler's Method

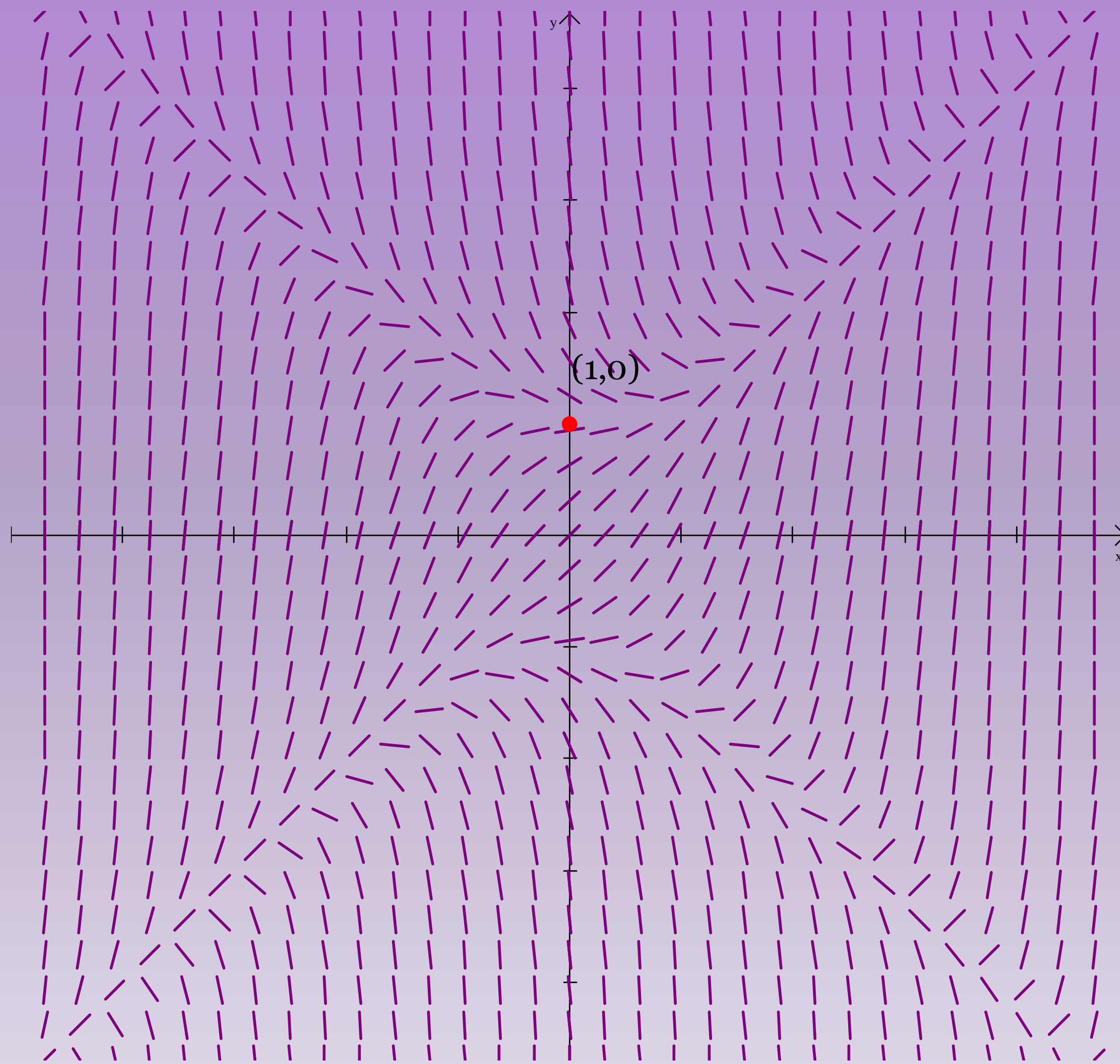
Use Euler's Method to find $y(1)$ if

$$\frac{dy}{dx} = x^2 - y^2 + 1$$

and $y(0) = 1$ using a step size of ¼ and a step size of 0.1.

Euler's Method

$$\frac{dy}{dx} = x^2 - y^2 + 1$$



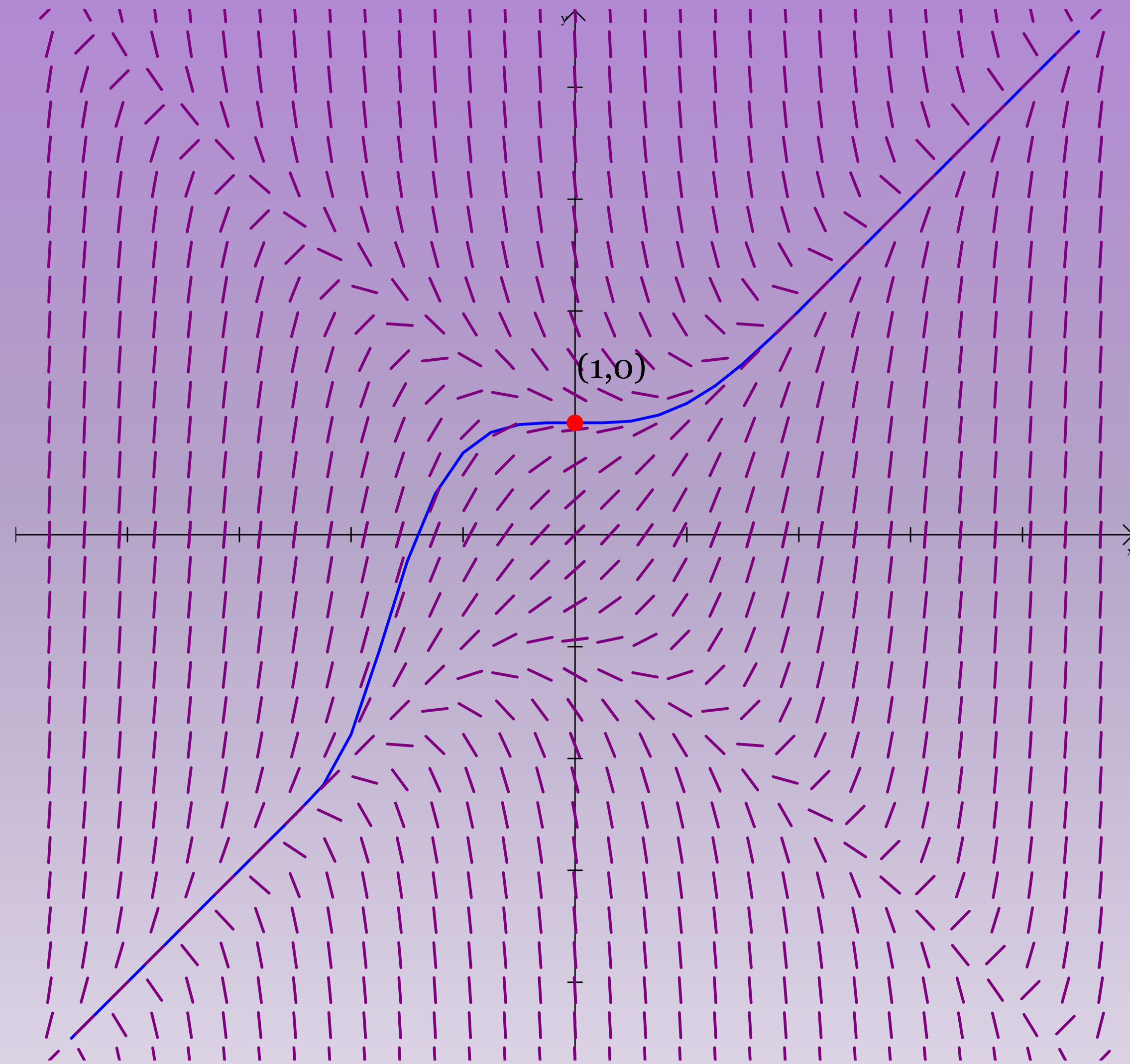
Euler's Method

$$\frac{dy}{dx} = x^2 - y^2 + 1$$

x_{old}	y_{old}	m	Δx	Δy	x_{new}	y_{new}
0.00	1.00	0.00	0.25	0.00	0.25	1.00
0.25	1.00	0.0625	0.25	0.15625	0.50	1.15625
0.50	1.15625	-0.274414	0.25	-0.0686035	0.75	1.0876465
0.75	1.0876465	0.379525	0.25	0.0948812	1.00	1.1825277
1.00	1.1825277					

Euler's Method

$$\frac{dy}{dx} = x^2 - y^2 + 1$$



Euler's Method

$$\frac{dy}{dx} = x^2 - y^2 + 1$$

x_{old}	y_{old}	m	Δx	Δy	x_{new}	y_{new}
0.0	1.00	0.00	0.1	0.00	0.1	1.00
0.1	1.00	0.01	0.1	0.001	0.2	1.001
0.2	1.001	0.037999	0.1	0.0037999	0.3	1.0048
0.3	1.0048	0.080377	0.1	0.0080377	0.4	1.012837
0.4	1.012837	0.13416	0.1	0.013416	0.5	1.026254
0.5	1.026254	0.196803	0.1	0.0196803	0.6	1.045934
0.6	1.045934	0.266022	0.1	0.0266022	0.7	1.072536
0.7	1.072536	0.339666	0.1	0.0339666	0.8	1.1065027
0.8	1.106502	0.415651	0.1	0.0415651	0.9	1.1480679
0.9	1.148067	0.49194	0.1	0.049194	1.0	1.197262
1.0	1.197262					

Euler's Method

$$\frac{dy}{dx} = x^2 - y^2 + 1$$