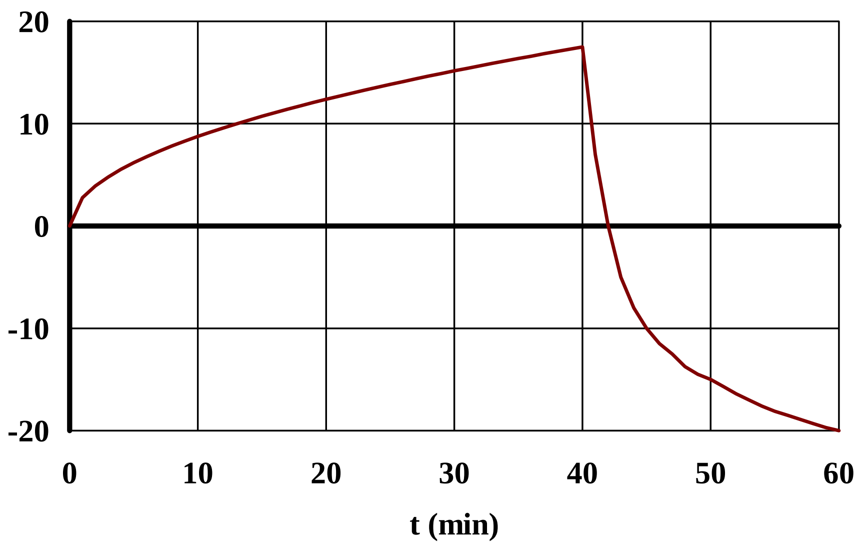
Spring 2009 MATH 6102

Derivatives, Graphically

The Montgolfier brothers (Joseph and Etienne) were 18th century pioneers in the field of hot-air ballooning. Had they had appropriate instruments, they might have left us a record of one of their early experiments, like the flight shown below. The graph is of their vertical velocity, *v*, with upwards as positive.



(a) Over what intervals was the acceleration positive? Negative? Zero?

(b) What was the greatest altitude achieved, and at what time?

(c) At what time was the magnitude of the acceleration greatest? What was the direction of their acceleration at that time?

(d) What might have happened during this flight to explain the answer to (c)?

(e) This particular flight ended on top of a hill. How do you know that it did, and what was the height of the hill above the starting point?

Graphing and Derivatives

Draw graphs which describe the following conditions. Some of these descriptions have been collected from news items.

1. An economist said, "*The rate of increase has decreased.*"

2. I got on the elevator at the ground floor and went up to the tenth floor without stopping. Draw the graph of acceleration. Draw the graph of speed. Draw the graph of position.

3. The truck was accelerating faster, but the car was going faster.

4. The economy has bottomed out.

5. Sketch a graph of a function f over the coordinate axis with domain [0,6]. Now, graph the integral of f on $[0,x]$. How does this integral differ from the antiderivative?

A Visual Definition of ***Differentiable***

A function is differentiable at any point where it is locally linear. To see whether a function is locally linear at a point, zoom in on the graph using a calculator. If the curve appears linear in the neighborhood of a point, then the function is differentiable at that point.

1. Graph $y = x^2$ on your graphing calculator. Zoom in on a neighborhood of $(1,1)$ until your graph appears linear. Find the slope of this segment of your graph by finding points using the trace and the definition of slope. How does this value compare to the derivative by formula?

This investigation suggests that properties of the slope of a linear function should tell you properties of derivatives of arbitrary functions.

2. For any linear function, $f(x) = mx + b$, find its inverse and the slope of its inverse. What does this suggest about the derivatives of inverse functions?
3. For any linear functions, $f(x) = mx + b$ and $g(x) = nx + d$, find the function $f(g(x))$. Find the slope of the composition of the linear functions. What does this suggest for the derivative of a composition of functions?
4. Find the slopes of products and quotients of linear functions. Note: This should illustrate that you cannot expect the derivative of a product to be the product of the derivatives or the derivative of a quotient to be the quotient of the derivatives.

   Let's consider the two linear functions, $f(x) = mx + b$ and $g(x) = nx + d$. Their product is

$$\begin{aligned} h(x) &= f(x) \cdot g(x) \\ &= (mx + b)(nx + d) \\ &= mnx^2 + mbx + ndx + bd \end{aligned}.$$

   The best linear approximation to this parabola is

$$\begin{aligned} 2mnx + md + nb &= m(nx + d) + n(mx + b) \\ &= f'(x) \cdot g(x) + g'(x) \cdot f(x) \end{aligned}$$

   and this is our Product Rule. Local linearity indicates that this should hold for all functions.

5. Draw the graph of $y = \sin(x)$ and use your idea of the derivative to sketch its derivative. Now draw the graph of $y = \sin(2x)$ and sketch its derivative. How does the 2 in the second function affect the graph of its derivative? How does this observation relate to your observation in Problem 3?

Newton's Method

Many problems in mathematics require finding a root of an equation. There are several techniques for finding these roots. Rarely, in practice, is the function nice enough to yield to gentle pressure and cough up its roots algebraically. Quadratic equations are ones that do behave nicely and, even though we know that there are general solutions for cubic and quartic equations, we often don't remember them. Modern Algebra, via Galois and Abel, assures us that there is no general form for finding the roots of a general polynomial of degree 5 or larger by radicals.

Most of our techniques are then numerical in nature. We have already discussed the bisection method. It is simple and usually quite effective. There are two major drawbacks:

- *It cannot locate a root where the curve is tangent to, but does not cross, the x-axis.*
- *It is slow—requiring a large number of iterations to converge to the root. If this is to be done only once, then this might not be too onerous. However, when we need to solve an equation a number of times for a different value of the parameter, this can be very slow.*

Suppose we have $y = f(x)$. *Let the equation* $f(x) = 0$ *have a root at r. Since linear equations are easy to solve and since we know that nice functions are locally linear, we might expect the best linear approximation to the function at a point to help us solve for this root. We will need a place to start, a place at which we can find the tangent line to the curve and a place that is close to the actual root,* r. *Let's take* x_0 *as our first approximation to* r. *The equation of the tangent line at* $x = x_0$ *is* $y = f(x_0) + f'(x_0)(x - x_0)$. *Since we are looking for a root, we need to know when* $y = 0$, *so we set* $y = 0$ *in the above equation and solve for* x. *We get:*

$$x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

This is now a better estimate for r, *and calling this number* x_1, *we will start the process again. We will end up with the following recursion relation:*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

provided that $f'(x_n) \neq 0$. *This process converges very quickly when it converges. It requires:*

- we know the derivative of the function $f(x)$;
- $f'(x_n) \neq 0$.

To help with the first problem, we can use the numerical derivative of the function given to us by the difference quotient, or the symmetric difference quotient. We also have the numerical derivative available to us on the calculator— NDeriv(, under the MATH menu (MATH 8).
We can enter our function as Y1. Then, we put

Y2 = X - Y1 / NDeriv(Y1,X,X)

We begin by evaluating Y2 at our guess for the root, and then iterate the process. This is simple with the TI-84.

Let's find the first non-zero root of $f(x) = \sin(x)$, with our first guess being $x_0 = 3$.
First, we put:

Y1 = sin(X)
Y2 = X - Y1 / NDeriv(Y1,X,X)

Then on the HOME screen we type:

```
Y2(3)
        3.142546567
Y2(Ans)
        3.141592653
Y2(Ans)
        3.141592654
Y2(Ans)
        3.141592654
```

This is not as hard as you might imagine. To get Y2(3) the keystrokes are

1 (Function),
2 (Y2)

Then, to get Y2(Ans) the keystrokes are:

The remaining entries are given by

If x_n is an estimate for the solution to $f(x) = 0$, then the next (better) estimate is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

1. Graph $y = x^3 - 6x^2 + 4x + 1$.

2. Use Newton's Method to find the middle zero of the function with error < 0.01.

3. What values could have been selected as the initial value of x to find the zero requested? Explain how you know which values can be selected as the starting point?

4. From your starting point how many iterations were required to guarantee the accuracy requested?

HINT: Calculate the slope of $y = x^3 - 6x^2 + 4x + 1$ at the value you have found for your approximate solution.

5. Use Newton's Method to find the largest value for x for which $\sin(x) = \ln(x)$.

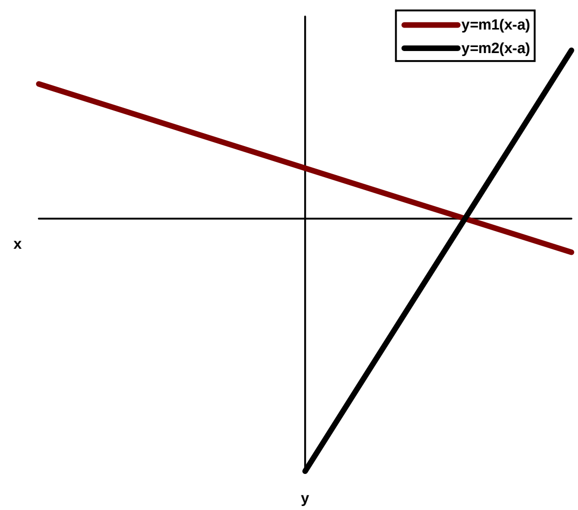
l'Hospital's Rule

We don't need to hold l'Hospital's Rule until 2nd semester calculus. We can think of it as an application of the derivative. As such we can give a plausible argument for why it will work, not a formal proof.

Let's start our study with linear functions. Suppose:

$$f(x) = m_1(x-a)$$
$$g(x) = m_2(x-a)$$

So, f and g are just two lines with a common x-intercept at $x = a$. Then the graphs of $y = f(x)$ and $y = g(x)$ around $x = a$ look like



Now $\lim_{x\to a}\frac{f(x)}{g(x)}$ has indeterminate form $0/0$. For $x \neq a$,

$$\frac{f(x)}{g(x)} = \frac{m_1(x-a)}{m_2(x-a)} = \frac{m_1}{m_2}$$

Thus,

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{m_1}{m_2}$$

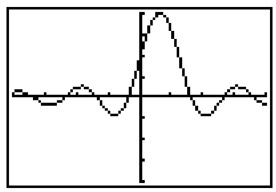
So the limit of the quotient of two linear functions at $x = a$ is the quotient of their slopes.

Now, what happens if $f(x)$ and $g(x)$ are two functions that are differentiable except possibly at $x = a$ and $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$. We now appeal again to the concept of local linearity. Since $f(x)$ and $g(x)$ are differentiable around $x = a$, we know that when we zoom in on their graphs near $x = a$ we should get something that looks linear for each of them.

Let's look at an example. What is

$$\lim_{x\to 1}\frac{2\sin(2x-2)}{x-1}?$$

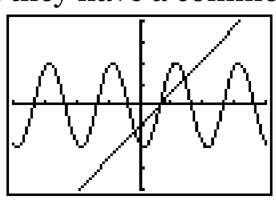
Immediately, we recognize that we get the indeterminate form 0/0 when we substitute $x = 0$. When we plot this function in the ZTrig window on the TI-84 we get a picture that indicates that the limit should exist and might be 4.



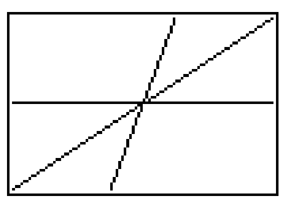
$-2\pi \le x \le 2\pi$

$-4 \le y \le 4$

Now, we will plot the two separate functions, $f(x) = 2\sin(2x-2)$ and $g(x) = x-1$ over this same range. We know that they have a common root at $x = 1$.



Let's look at them closer. In the following picture we have zoomed in to the rectangle $0.9 \le x \le 1.1$ and $-0.1 \le y \le 0.1$:



Now, our graphs look like the intersections of two lines. From what we saw above we recognize that the limit should be quotient of the two slopes. We recognize that in the limit these two slopes are $f'(a)$ and $g'(a)$.

Thus, we get

l'Hospital's Rule

If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$

and $\lim_{x\to a} \dfrac{f'(x)}{g'(x)} = L$,

then $\lim_{x\to a} \dfrac{f(x)}{g(x)} = L$

Derivative Problems

1. Census figures for the U.S. population (in millions) are listed in the following table. Since population varies with time, there is a function, *f*, such that $P = f(t)$. Assume that *f* is increasing (as the values in the table suggest). Then *f* is invertible.

   (a) What is the meaning of $f^{-1}(100)$?

   (b) What does the derivative of $f^{-1}(P)$ at $P = 100$ represent? What are its units?

   (c) Estimate $f^{-1}(100)$.

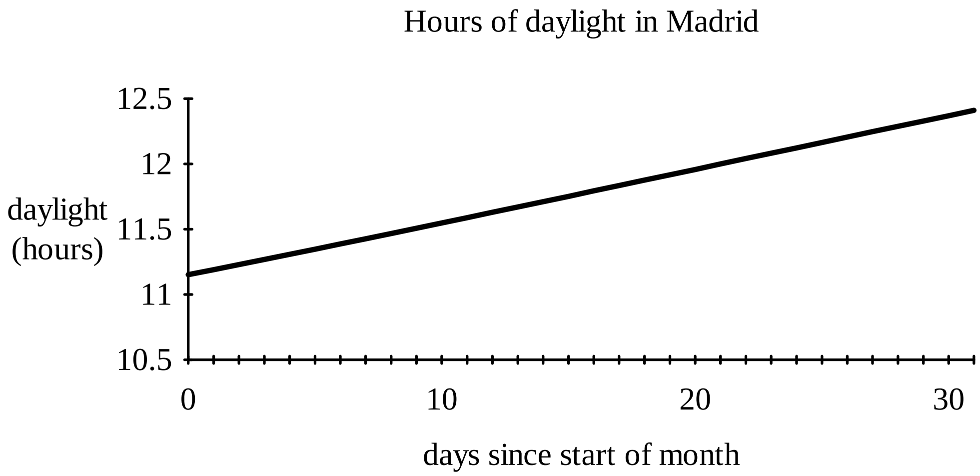
   (d) Estimate the derivative of $f^{-1}(P)$ at $P = 100$.

Year	Population	Year	Population
1790	3.9	1900	76.0
1800	5.3	1910	92.0
1810	7.2	1920	105.7
1820	9.6	1930	122.8
1830	12.9	1940	131.7
1840	17.1	1950	150.7
1850	23.1	1960	179.0
1860	31.4	1970	205.0
1870	38.6	1980	226.5
1880	50.2	1990	248.7
1890	62.9	2000	281.4

2. The number of hours, *H*, of daylight in Madrid as a function of date is approximated by the formula

$$H = 12 + 2.4\sin(0.0172(t - 80))$$

   where *t* is the number of days since the start of the year. The following figure shows a one-month portion of the graph of *H* against *t*.



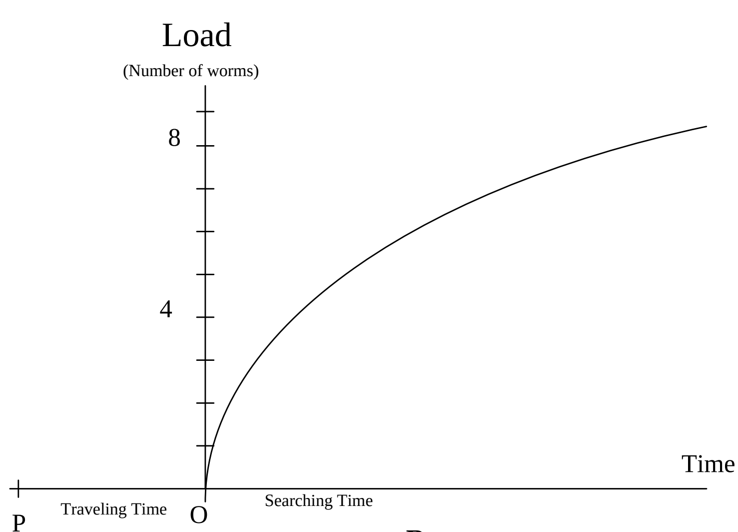
(a) Comment on the shape of the graph. Why does it look like a straight line?

(b) What month does this graph show? How do you know?

(c) What is the approximate slope of this line? What does the slope represent in practical terms?

3. A bird such as a starling feeds worms to its young. To collect worms, the bird flies to a site where worms are to be found, picks up several in its beak, and flies back to its nest. The *loading curve* in the following figure shows how the number of worms (the load) a starling collects depends on the time it has been searching for them. The curve is concave down because the bird can pick up worms more efficiently when its beak is empty; when its beak is alredy partly full, the bird becomes much less efficient. The traveling time (from nest to site and back) is represented by the distance *PO* in the figure. Suppose the bird wants to maximize the rate at which it brings worms to the nest, where

$$\text{Rate Worms Arrive at Nest} = \frac{\text{Load}}{\text{Traveling Time} + \text{Searching Time}}$$

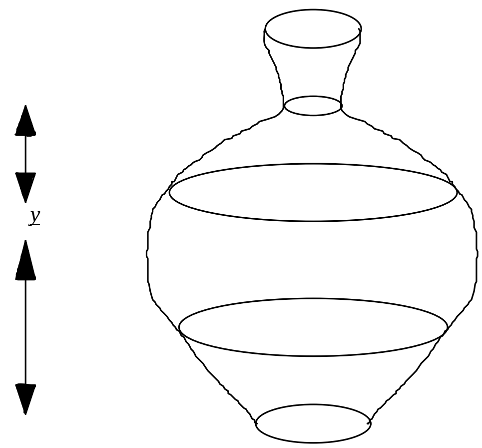


(a) Draw a line in the figure whose slope is this rate.

(b) Using the graph, estimate the load which maximizes this rate.

(c) If the traveling time is increased, does the optimal load increase or decrease? Why?

4. Consider the vase in the following figure. Assume the vase is filled with water at a constant rate, (*i.e.*, constant volume per unit time)



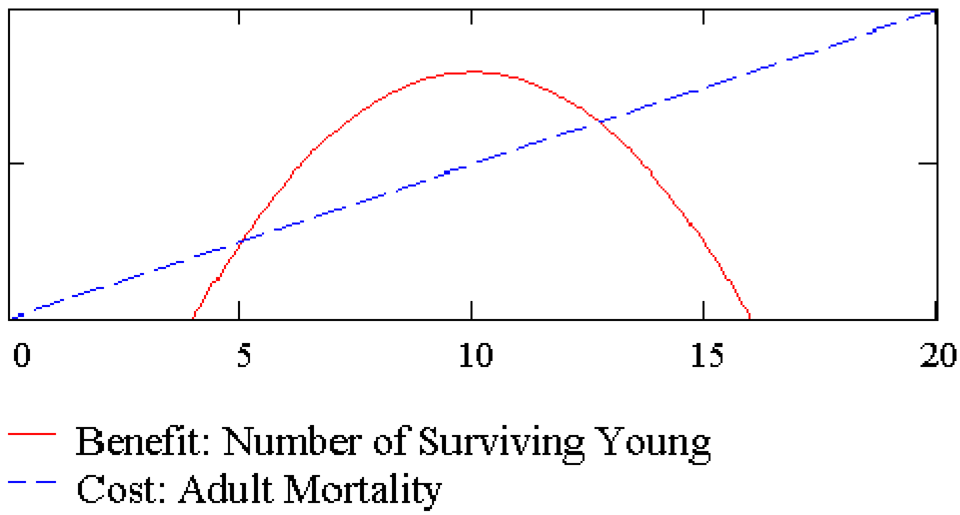
(a) Graph $y = f(t)$, the depth of the water, against time, t. Pay attention to the concavity. Mark interesting points on the graph.

(b) Where does $y = f(t)$ grow fastest, slowest? Estimate the ratio between these two growth rates.

5. When birds lay eggs, they do so in clutches of several at a time. When the eggs hatch, each clutch gives rise to a brood of baby birds. We want to determine the clutch size which maximizes the number of birds surviving to adulthood per brood. If the clutch is small, there are few baby birds in the brood; if the clutch is large, there are so many baby birds to feed that most die of starvation. The number of surviving birds per brood as a function of clutch size is shown in the figure on the next slide.

(a) Estimate the clutch size which maximizes the number of survivors per brood.

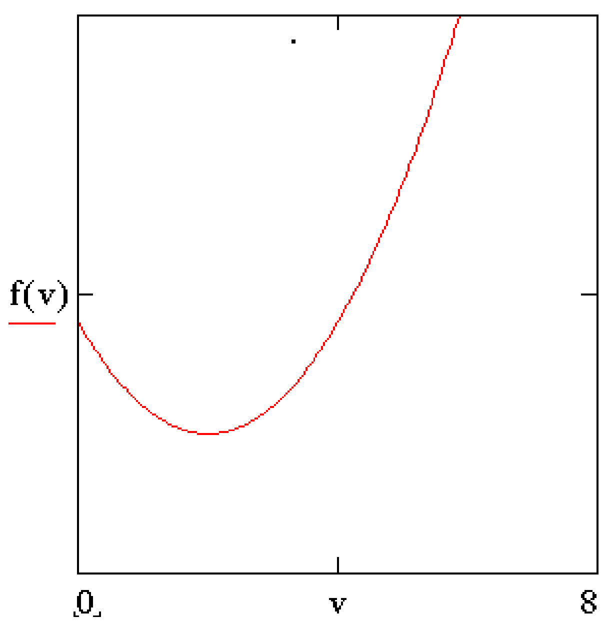
(b) Suppose also that there is a biological cost to having a larger clutch: the female survival rate is reduced by large clutches. This cost is represented by the dotted line in the following figure. If we take cost into account by assuming that the optimal clutch size in fact maximizes the vertical distance between the curves, what is the new optimal size?



The horizontal axis is the Clutch Size and the vertical axis is the Benefit or Cost.

6. A cat, walking along the window ledge of a New York apartment, knocks off a flower pot which falls to the street 200 feet below. How fast is the flower pot traveling when it hits the street? Give your answer in ft/sec and in mph.

7. Let *f(v)* be the energy consumption of a flying bird, measured in joules per second (a joule is a measure of energy), as a function of its speed *v* (in meters/sec).



(a) Explain the shape of this graph (in terms of the way birds fly)

Now let *a(v)* be the energy consumption of the same bird, measured in joules *per meter*.

(b) What is the relationship between *f(v)* and *a(v)*?

(c) Where is *a(v)* minimal?

(d) Should the bird try to minimize *f(v)* or *a(v)* when it is flying? Why?

8. Suppose you put a yam in a hot oven, maintained at a constant temperature of 200°C. As the yam picks up heat from the oven, its temperature rises.
   (a) Draw a possible graph of the temperature T of the yam against time t (minutes) since it is put into the oven. Explain any interesting features of the graph, and in particular explains its concavity.
   (b) Suppose that, at $t = 30$, the temperature T of the yam is 120° and increasing at the (instantaneous) rate of 2°/min. Using this information, plus what you know about the shape of the T graph, estimate the temperature at time $t = 40$.
   (c) Suppose in addition I tell you that at $t = 60$, the temperature of the yam is 165°. Can you improve your estimate of the temperature at $t = 40$?
   (d) Assuming all the data given so far, estimate the time at which the temperature of the yam is 150°.

Applications of the Derivative
Maxima and Minima

1. Find two nonnegative numbers whose sum is 9 and so that the product of one number and the square of the other number is a maximum. *(3 and 6)*
2. Build a rectangular pen with three parallel partitions using 500 feet of fencing. What dimensions will maximize the total area of the pen? (50×125)
3. An open rectangular box with square base is to be made from 48 ft.^{2} of material. What dimensions will result in a box with the largest possible volume? $(4 \times 4 \times 2)$
4. A container in the shape of a right circular cylinder with no top has surface area 3π ft.^{2}. What height h and base radius r will maximize the volume of the cylinder? $(r = h = 1)$
5. A sheet of cardboard 3 ft. by 4 ft. will be made into a box by cutting equal-sized squares from each corner and folding up the four edges. What will be the dimensions of the box with largest volume? $\left(\frac{7-\sqrt{13}}{6}\right)$
6. Consider all triangles formed by lines passing through the point (8/9, 3) and both the x- and y-axes. Find the dimensions of the triangle with the shortest hypotenuse. $(x = 26/9, y = 13/3)$

7. Find the point (x, y) on the graph of $y = \sqrt{x}$ nearest the point (4, 0). $\left(x = 7/2,\ y = \sqrt{7/2}\right)$

8. A cylindrical can is to hold 20 π m.^{3} The material for the top and bottom costs \$10/m.^{2} and material for the side costs \$8/m.^{2} Find the radius *r* and height *h* of the most economical can. (*r* = 2, *h* = 5)

9. You are standing at the edge of a slow-moving river which is one mile wide and wish to return to your campground on the opposite side of the river. You can swim at 2 mph and walk at 3 mph. You must first swim across the river to any point on the opposite bank. From there walk to the campground, which is one mile from the point directly across the river from where you start your swim. What route will take the least amount of time? (*If x is the distance from the point across the river and the campground to which you will swim, then the minimum occurs when* $x = 2/\sqrt{5}$.)

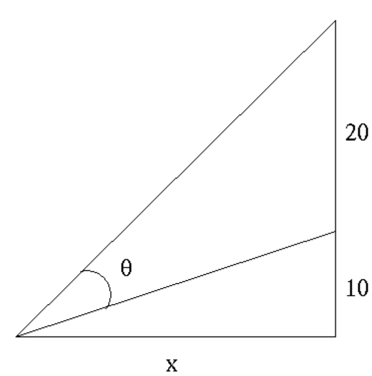
10. Construct a window in the shape of a semi-circle over a rectangle. If the distance around the outside of the window is 12 feet, what dimensions will result in the rectangle having largest possible area? (*Width* = $\frac{12}{2+\pi}$, *height* = 3)

11. There are 50 apple trees in an orchard. Each tree produces 800 apples. For each additional tree planted in the orchard, the output per tree drops by 10 apples. How many trees should be added to the existing orchard in order to maximize the total output of trees? (*15 additional trees*)

12. Find the dimensions of the rectangle of largest area which can be inscribed in the closed region bounded by the *x*-axis, *y*-axis, and graph of $y = 8 - x^3$. ($x = \sqrt[3]{2}$, , $y = 6$)

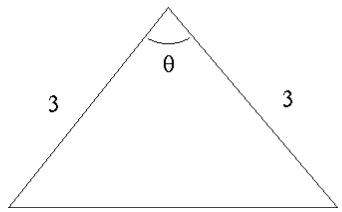
13. Consider a rectangle of perimeter 12 inches. Form a cylinder by revolving this rectangle about one of its edges. What dimensions of the rectangle will result in a cylinder of maximum volume? (*r* = 4, *h* = 2)

14. A movie screen on a wall is 20 feet high and 10 feet above the floor. At what distance *x* from the front of the room should you position yourself so that the viewing angle θ of the movie screen is as large as possible? (See diagram.) ($x = 10\sqrt{3},\ \theta = \frac{\pi}{6}$)



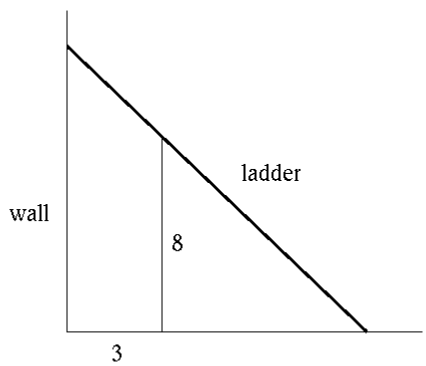
15. Find the dimensions (radius r and height h) of the cone of maximum volume which can be inscribed in a sphere of radius 2. ($r = 4\sqrt{2}/3,\ h = 8/3$)

16. What angle θ between two edges of length 3 will result in an isosceles triangle with the largest area? (See diagram.) $(\theta = \pi/2)$



17. Of all lines tangent to the graph of $y = \dfrac{6}{x^2+3}$, find the tangent lines of minimum slope and maximum slope. (Maximum slope ¾ and minimum slope –¾)

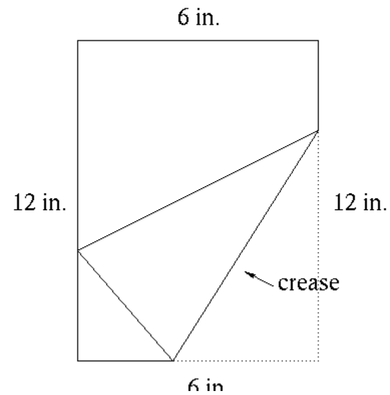
18. Find the length of the shortest ladder that will reach over an 8-ft. high fence to a large wall which is 3 ft. behind the fence. (See diagram.) (L = 17.64)



19. Find the point $P = (x, 0)$ on the x-axis which minimizes the sum of the squares of the distances from P to (0, 0) and from P to (3, 2). ($x = 3/2$)

20. Car B is 30 miles directly east of Car A and begins moving west at 90 mph. At the same moment car A begins moving north at 60 mph. What will be the minimum distance between the cars and at what time t does the minimum distance occur? (t = 13 *min* 48 *secs*, d = 24.96)

21. A rectangular piece of paper is 12 inches high and six inches wide. The lower right-hand corner is folded over so as to reach the leftmost edge of the paper (See diagram.).



Find the minimum length of the resulting crease. (L = $9\sqrt{3}/2$)

22. My neighbors have a very loud stereo. The volume knob turns half a circle (angles θ between 0 and $\pi/2$) and the volume of the music (usually Prince) is given by the function $V(\theta)=110\sin(\theta/2)$ decibels (dB). One night at 3:30 in the morning I notice the lyrics "I wanna be your fantasy, and you can be mine..." increasing from a volume of 88 dB at a rate of 1 decibel per second! At what rate can I deduce that my neighbor is turning his volume knob? (1/33 *radians per second*)

23. Art and Val are on their annual hunting trip. This time, however, **they** plan on outsmarting the deer! Val is sitting to Art's right (east) when the perfect buck appears 40 meters to the north. Art aims the gun, but Val sneezes. The deer startles and takes off straight southeast at 30 meters per second. Art turns to keep the deer centered in his sight, but can't get a clean shot. At the instant Art smacks Val in the head with the barrel of the gun, how fast was he rotating?

24. Brian got a new sand box over the weekend and took his "friends," He-Man and Skeletor, out to play make-believe. Skeletor tied He-Man to a pole and began dumping sand on top of him at a rate of 4 cubic inches per second. He-Man is six inches tall. At the moment he is half buried, He-Man notices that the sand is rising at a rate of 1/2 inch per second. How much longer does He-Man have to come up with a way to escape before he is completely buried?

25. Eleanor and Regina operate a tour service for people who would like to travel from Austin to Shreveport to go riverboat gambling. The minimum size for a group is 50 people at $200 per person (it includes food, transportation, and a seminar on the mathematics of gambling). For each additional person, up to a maximum of 80 people total, *everyone's* fare is reduced by $2. Given that it costs them $6000 (a fixed cost for their offices, the buses, licenses, *etc.*) plus $32 per person to conduct the tour, how many people does it take to maximize their profits?

    What if property taxes increase, and their fixed costs rise to $7000? What if that happens **and** the Louisiana state legislature puts an $8 per person tax on gambling tours?

26. Max has decided to try to get on Dr. Davis's good side by building a life-size origami statue of her as a gift. He begins with a sheet of paper that is 40' long and 60'wide, and wants the first fold to be particularly special. Specifically, the bottom right corner is to be folded to a point on the left side so that the length of the crease is a minimum. To what point should Max fold the corner to achieve this incredibly symbolic feat?

27. Brandie and Bat-Girl show up at Brian's new sand box. Bat-Girl says that burying He-Man is boring and she would rather do Calculus! Help Bat-Girl and Skeletor do the following problem from one of Kathy's old exams:

   a. Let $f(x)=\sin(x)+\cos(x)$ on the interval $[0,2\pi]$.
      i) Differentiate and simplify.
      ii) Find all points *c* where $f'(c)=0$ or $f'(c)$ does not exist.
      iii) Use the first derivative test to classify the above points.
      iv) Find all local and global extrema.

28. Inspired by recently seeing *Saturday Night Fever* Melanie is redecorating her 16'×12' dorm room in a disco theme. Her roommate opens the door and is shocked by the sight of a disco ball rotating once every 2 seconds from the center of the ceiling. Her horror is replaced by a trance-like state as she is hypnotized from tracking one of the spots of light spinning around the room. As this spot of light enters a corner going from a long wall to a short wall, how fast is it moving?

29. Find the shortest line segment with endpoints on the x- and y-axes going through the point (1,8).

30. A garden is designed to be in the shape of a circular sector with radius R and angle θ. If the area A is to be a constant, find the dimensions (R, θ) which minimize the length of fence around the perimeter.

31. A sector with angle φ is cut from a circle of radius 12 inches, and the resulting edges are brought together to form a cone. Find the magnitude of φ so that the volume of the cone is maximized.

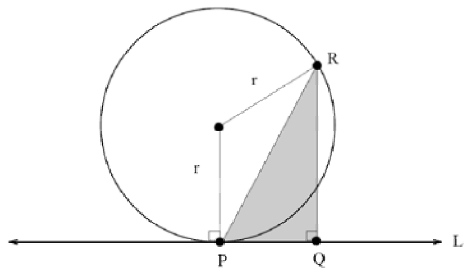
32. Twenty feet of wire are to be used to form two figures. In each of the following cases, how much should be used for each figure so that the total enclosed area is a maximum?

   a) equilateral triangle and square
   b) square and regular pentagon
   c) regular pentagon and hexagon
   d) regular hexagon and circle

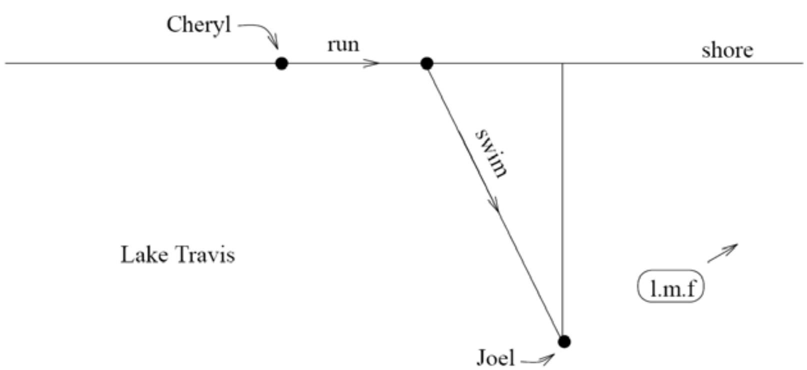
   (Hint: The area of a regular polygon with *n* sides of length *x* is $A=\frac{n}{4}\cot\left(\frac{\pi}{n}\right)x^2$).

   What can you conclude from this pattern?

33. Suppose you are given a circle of radius *r* and a tangent line *L* to the circle through a point *P* on the circle. From a variable point *R* on the circle, a perpendicular *PQ* is drawn to *L* with *Q* on *L*. Determine the maximum of the area of triangle *PQR*.



34. Joel has fallen off of his "Little Mermaid" floaty 200 feet from shore in Lake Travis. He cannot swim. Cheryl is at a point 200 feet down the shore from the point closest to Joel. She can run 18 ft/s and can swim at a rate of 5 ft/s.



a) To what point on the shore should she run before diving into the lake **if** she wants to reach Joel as quickly as possible?
b) Once Joel falls into the water, he can manage to thrash about for one minute. It takes Cheryl 10 seconds to notice Joel is in trouble. Can she reach him in time?