Differential Forms and Integration

1 Differential Forms

Let U be an open set in$\mathbf{R}^3$ and assume that the functions we are discussing are differentiable as many times as needed. The forms and the functions being discussed are all differentiable, so we will drop that adjective, since it is to be applied to everything in this section.

Definition 1 *A 0-form on U is a real-valued function $f\colon U \to \mathbf{R}$. Two 0-forms f_1 and f_2 on U can be added or multiplied giving 0-forms $f_1 + f_2$ and $f_1 \cdot f_2$. A zero 0-form is the constant function $f(x, y, z) = 0$.*

Definition 2 *The formal expressions dx, dy, and dz are called basic 1-forms. A 1-form on the set U is a combination*

$$\alpha = f(x,y,z)dx + g(x,y,z)dy + h(x,y,z)dz = f\,dx + g\,dy + h\,dz,$$

where the coefficients (or components) $f(x,y,z)$, $g(x,y,z)$, $h(x,y,z)$ are real-valued functions defined on U.

For the time being we will not attach any meaning to dx, dy, and dz. Think of them as forming a basis of a three-dimensional vector space, where the real-valued functions play the role of scalars. We say that a 1-form is a linear combination of basic 1-forms dx, dy, and dz. The order of dx, dy, and dz in a 1-form is not relevant. The zero 1-form is the 1-form $0 = 0dx + 0dy + 0dz$.

Let $\alpha = x^2dx + xydy - yzdz$ and $\beta = y^2dx - yzdy + dz$. Then $\alpha + \beta = (x^2 + y^2)dx + (xy - yz)dy + (1 - yz)dz$. In general, if $\alpha = f_1dx + f_2dy + f_3dz$ and $\beta = g_1dx + g_2dy + g_3dz$ and p is a real-valued function, then

$$\alpha + \beta = (f_1 + g_1)dx + (f_2 + g_2)dy + (f_3 + g_3)dz$$

and

$$p\alpha = pf_1dx + pf_2dy + pf_3dz.$$

Definition 3 *The expressions $dxdy$, $dydz$, and $dzdx$ are called basic 2-forms. A 2-form on the set U is a combination*

$$\beta = f(x,y,z)dxdy + g(x,y,z)dydz + h(x,y,z)dzdx,$$

where the coefficients are real-valued functions.

Again, think of the basic 2-forms as forming a basis for a three-dimensional vector space. A 2-form is a linear combination of the basic 2-forms. The order of the basic 2-forms in the 2-form does not matter. Just as for 1-forms, we can add 2-forms and multiply by a scalar function.

If $\alpha = f_1dxdy + f_2dydz + f_3dzdx$ and $\beta = g_1dxdy + g_2dydz + g_3dzdx$ and p is a real-valued function, then

$$\alpha + \beta = (f_1 + g_1)dxdy + (f_2 + g_2)dydz + (f_3 + g_3)dzdx$$

and

$$p\alpha = pf_1dxdy + pf_2dydz + pf_3dzdx.$$

The zero 2-form is the 2-form $0 = 0dxdy + 0dydz + 0dzdx$.

We never add forms of different degrees, such as a 1-form and a 2-form.

Definition 4 *The basic 3-form is the expression $dxdydz$. A 3-form on U is an expression*

$$\eta = f(x,y,z)dxdydz,$$

where f is a scalar function.

This time there is only one basic 3-form, so the vector space is 1-dimensional. If $\eta = fdxdydz$ and $\gamma = gdxdydz$ and p is a scalar function, then

$$\eta + \gamma = (f + g)dxdydz \qquad \text{and} \qquad p\eta = pfdxdydz.$$

The zero 3-form is $0 = 0dxdydz$.

2 Wedge Product

The *wedge product* is an operation on forms that satisfies the following properties:

(a) If α is a m-form and β is an n-form, $0 \le n, m \le 3$ then their wedge product is an $(n + m)$-form $\alpha \wedge \beta$.

(b) Anticommutativity: $\alpha \wedge \beta = (-1)^{nm}\beta \wedge \alpha$, where α is a m-form and β is an n-form.

(c) Associativity: $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$, for any forms α, β, and γ.

(d) If 0 is a zero form of any degree, then $\alpha \wedge 0 = 0$ for any form α.

(e) If f is a 0-form (a scalar function) then the wedge product $f \wedge \alpha$ is just the product $f \cdot \alpha$ of a form and a function: $f \wedge \alpha = f \cdot \alpha$.

(f) Distributivity: if α and β are of the same degree, and γ is any form, then $(\alpha + \beta) \wedge \gamma = \alpha \wedge \gamma + \beta \wedge \gamma$.

(g) Homogeneity with respect to scalar functions: if f is a 0-form, then $f(\alpha \wedge \beta) = (f\alpha) \wedge \beta = \alpha \wedge (f\beta)$.

(h) The wedge products of basic 1-forms are basic 2-forms:

$$dx \wedge dy = dxdydy \wedge dz = dydzdz \wedge dx = dzdx.$$

(i) The basic 3-form is the wedge product of basic 1-forms in their cyclic order:

$$dx \wedge dy \wedge dz = (dx \wedge dy) \wedge dz = dx \wedge (dy \wedge dz) = dxdydz.$$

We can derive a few more simple properties from these. From anticommutativity we get

$$dydx = dy \wedge dx = (-1)^{1 \cdot 1} dx \wedge dy = -dx \wedge dy = -dxdy.$$

Also,

$$dx \wedge dx = -dx \wedge dx$$

from anticommutativity. Thus,

$$dx \wedge dx + dx \wedge dx = 2dx \wedge dx = 0.$$

Therefore, $dx \wedge dx = 0$. It follows that $dy \wedge dy = 0$ and $dz \wedge dz = 0$. Likewise a triple product in which a basic 1-form appears more than once must be zero. For example,

$$\begin{aligned} dxdy \wedge dx &= (dx \wedge dy) \wedge dx = (-dy \wedge dx) \wedge dx \\ &= -dy \wedge (dx \wedge dx) = -dy \wedge 0 = 0. \end{aligned}$$

Since there are only three basic 1-forms in $\mathbf{R}^3$, every 4-form (written as a wedge product of 1-forms) has at least one of dx, dy, or dz repeated — and is therefore zero. In other words, all forms of degree greater than three in $\mathbf{R}^3$ are zero.

Lemma 1 *For any form* α *in* $\mathbf{R}^3$, $\alpha \wedge \alpha = 0$.

The proof of this lemma is a brute force calculation using the above properties.

Example 1 Let $\alpha = ydxdy + xzdydz$ and $\beta = dx + ydy + z^2dz$. Find $\alpha \wedge \beta$.

$$\begin{aligned}
\alpha \wedge \beta &= (ydxdy + xzdydz) \wedge (dx + ydy + z^2dz) \\
&= ydxdy \wedge dx + ydxdy \wedge ydy + ydxdy \wedge z^2dz + xzdydz \wedge dx + xzdydz \wedge ydy + szdydz \wedge z^2dz \\
&= yz^2dxdydz + xzdydz \wedge dx = yz^2dxdydz + (-1)^{2\times1}xzdx \wedge dydz \\
&= (yz^2 + xz)dxdydz
\end{aligned}$$

Definition 5 *The differential is an operation that assigns a $(k+1)$ form $d\alpha$ to a k-form α $(0 \leq k \leq 3)$ according to the following rules*

a) If $f\colon U \to \mathbf{R}$ is a 0-form, then df is the 1-form

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

b) If $\alpha = fdx + gdy + hdz$ is a 1-form, then $d\alpha$ is the 2-form

$$d\alpha = df \wedge dx + dg \wedge dy + dh \wedge dz,$$

where df, dg, and dh are computed by (a), since f, g, and h are scalar functions (or 0-forms).

c) If $\alpha = fdxdy + gdydz + hdzdx$ is a 2-form, the $d\alpha$ is the 3-form

$$d\alpha = df \wedge dxdy + dg \wedge dydz + dh \wedge dzdx.$$

d) If $\alpha = fdxdydz$ is a 3-form, then

$$d\alpha = 0.$$

Example 2 Compute the differential of each of the following forms.

a) $\alpha = e^{xyz}$.

$$\begin{aligned}
d\alpha &= \frac{\partial}{\partial x}(e^{xyz})dx + \frac{\partial}{\partial y}(e^{xyz})dy + \frac{\partial}{\partial z}(e^{xyz})dz \\
&= yze^{xyz}dx + xze^{xyz}dy + xye^{xyz}dz
\end{aligned}$$

b) $\alpha = (x^2 + y^2)dx + \sin(z)dz.$

$$\begin{aligned} d\alpha &= d(x^2+y^2)\wedge dx + d(\sin(z))\wedge dz \\ &= (2xdx + 2ydy + 0dz)\wedge dx + (\cos(z)dz)\wedge dz \\ &= 2ydy\wedge dx = -2ydxdy \end{aligned}$$

c) $\alpha = xdy$

$$d\alpha = d(x)\wedge dy = dxdy.$$

d) $\alpha = (x^2 + y^2)dxdz$

$$\begin{aligned} d\alpha &= d(x^2+y^2)\wedge dxdz \\ &= (2xdx + 2ydy)\wedge dxdz \\ &= 2ydydxdz = -2ydxdydz \end{aligned}$$

Now, our vector differential operators — gradient, curl, and divergence — can be interpreted as differentials on forms.

The differential of a 0-form f is given by

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = \nabla f\cdot d\mathbf{s},$$

where $d\mathbf{s} = dx\boldsymbol{\imath} + dy\boldsymbol{\jmath} + dz\boldsymbol{k}$ is a forma expression called the *line element*, and ∇f is the gradient of f.

Take a 1-form $\alpha = F_1dx+F_2dy+F_3dz$ and the associated vector field $\mathbf{F} = F_1\boldsymbol{\imath}+F_2\boldsymbol{\jmath}+F_3\boldsymbol{k}$. Computing the differential of α we get

$$d\alpha = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dx\wedge dy + \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)dy\wedge dz + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)dz\wedge dx$$

The coefficients of $d\alpha$ coincide with the components of curl($\mathbf{F}$). The coefficient of $dy\wedge dz$ is the $\boldsymbol{\imath}$ component of curl($\mathbf{F}$), the coefficient of $dz\wedge dx$ is the $\boldsymbol{\jmath}$ component of curl($\mathbf{F}$) and the coefficient of $dx\wedge dy$ is the $\boldsymbol{k}$ component fo curl($\mathbf{F}$).

Finally, let $\alpha = F_1dydz + F_2dzdx + F_3dxdy$ be a 2-form and the associated vector field $\mathbf{F} = F_1\boldsymbol{\imath} + F_2\boldsymbol{\jmath} + F_3\boldsymbol{k}$.

$$d\alpha = \left(pdF_1x + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)dxdydz = \mathrm{div}(\mathbf{F})dxdydz.$$

Theorem 1 *For k-forms α_1, α_2 and α and n-form β the following identities hold:*

a) $d(\alpha_1 + \alpha_2) = d\alpha_1 + d\alpha_2$.

b) $d(d\alpha) = 0$.

c) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$.

3 Forms and Integration

We have seen this before. Let $\alpha = F_1 dx + F_2 dy + F_3 dz$ be a 1-form on $\mathbf{R}^3$ and let $\sigma\colon [a, b] \to \mathbf{R}^3$ be a path. The *path integral of the 1-form α along σ* is defined by

$$\int_\sigma \alpha = \int_\sigma F_1 dx + F_2 dy + F_3 dz = \int_a^b \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right) dt.$$

Let $\alpha = F_1 dydz + F_2 dzdx + F_3 dxdy$ be a 2-form defined on U and let $\Phi(u, v) = (x(u, v), y(u, v), z(u, v))$ be a parametric representation of a surface $S \subseteq U$. Define the integral of the 2-form α over S, $\int_S \alpha$, as the double integral

$$\begin{aligned} \int_S \alpha &= \int_S F_1 dydz + F_2 dzdx + F_3 dxdy \\ &= \iint_D \left(F_1(\Phi(u, v)) \frac{\partial(y, z)}{\partial(u, v)} + F_2(\Phi(u, v)) \frac{\partial(z, x)}{\partial(u, v)} + F_3(\Phi(u, v)) \frac{\partial(x, y)}{\partial(u, v)} \right) dA \\ &= \iint_D \mathbf{F}(\Phi(u, v)) \cdot \mathbf{N}(u, v)\, dudv = \iint_S \mathbf{F} \cdot d\mathbf{S} \end{aligned}$$

where $\mathbf{F} = (F_1, F_2, F_3)$ and

$$\mathbf{N}(u, v) = \mathbf{T}_u(u, v) \times \mathbf{T}_v(u, v) = \left(\frac{\partial(y, z)}{\partial(u, v)}, \frac{\partial(z, x)}{\partial(u, v)}, \frac{\partial(x, y)}{\partial(u, v)} \right).$$

It follows that the integral of a 2-form $\alpha = F_1 dydz + F_2 dzdx + F_3 dxdy$ over a surface S is just the surface integral of the corresponding vector field $\mathbf{F} = (F_1, F_2, F_3)$.

Theorem 2 (Green's Theorem for Differential Forms) *Let $\alpha = Pdx + Qdy$ be a 1-form defined on a "good" region D with positively-oriented boundary $\sigma = \partial D$. Then*

$$\int_{\partial D} \alpha = \int_D d\alpha.$$

Theorem 3 (Gauss' Divergence Theorem for Differential Forms) *Assume that $V \subseteq \mathbf{R}^3$ is a "good(3D)" region with boundary ∂V oriented by an outward normal. Let α be a 2-form defined on an open set U containing V. Then*

$$\int_{\partial V} \alpha = \int_V d\alpha.$$

This is just a straightforward translation of Gauss' Divergence Theorem into the language of differential forms.

Theorem 4 (Stokes' Theorem for Differential Forms) *Let* $\alpha = F_1 dx + F_2 dy + F_3 dz$ *be a 1-form defined on an open set* U *in* $\mathbf{R}^3$. *Let* S *be an oriented surface parametrized by a one-to-one parametrization* $\Phi\colon D \to \mathbf{R}^3$ *and let* ∂S *be its positively-oriented piecewise smooth boundary curve. Then,*

$$\int_{\partial S} \alpha = \int_S d\alpha.$$

This is just a straightforward translation of Stokes' Theorem into the language of differential forms.

Note that all three of these theorems say the same thing!

$$\int_{\partial M} \alpha = \int_M d\alpha.$$